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A New Lomax Type Distribution: Properties, Copulas, Applications, Bayesian and Non-Bayesian Estimation Methods

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Abstract

A new three-parameter lifetime model called the odd Burr Lomax (OBLo) is defined and studied. The density of the OBLo model can be asymmetric heavy tail right skewed density and symmetric density with different useful shapes. Hazard Rate Function (HRF) of the odd Burr Lomax can be "monotonically decreasing", "**J**-hazard rate function", "increasing-constant" and "monotonically increasing". The approach of copula is used for deriving many bivariate odd Burr Lomax type distributions. Bayesian and non-Bayesian estimation methods are considered. Four non-Bayesian estimation methods are considered and compared such as the maximum likelihood estimation method, ordinary least square estimation method, weighted least square estimation method and Kolmogorov estimation method. The Bayesian estimation method is considered under the squared error loss function. We assessed the performance of the log-likelihood estimation method via simulation study. The odd Burr Lomax model could be chosen as the best model among Lomax, exponentiated Lomax, Kumaraswamy Lomax, Macdonald Lomax, beta Lomax, gamma Lomax, odd log-logistic Lomax, reduced odd log-logistic Lomax, reduced Burr-Hatke Lomax, reduced OBLo and special generalized mixture Lomax distribution in modeling the "failure times" and the "service times" data sets.

Keywords: Lomax model; maximum likelihood estimation; Simulations; Copulas; Renyi's entropy; Farlie Gumbel Morgenstern family; Bayesian estimation.

AMS Subjects Classification: 62N01; 62N02; 62E10.

0. Twin Statisticians: My Friends Bimal K. Sinha and Bikas K. Sinha by M. Masoom Ali

There is hardly any statistician of my generation or previous two or three generations who has not heard the names of the Sinha twins. Professor Bimal K. Sinha of University of Maryland Baltimore County, USA and his twin brother Professor Bikas K. Sinha of ISI Kolkata, India, have established themselves as equally famous and well-respected statisticians. I am not going to talk here about them as statisticians. I will rather talk about my personal relation with them as colleagues, friends and well-wishers.

My first meeting with the twin brothers was very interesting. In 1989 I went to the Aligarh Muslim University to present an invited paper at the International Symposium on Optimization and Statistics. During one of the conference sessions while I was listening to a speaker I just casually noticed one of the attendees and when I momentarily went out of the room I found the same person outside the room. I quickly went back to the room and found that person still sitting in the room listening to the lecture. When I went outside again the same person was still outside. I was wondering if I lost my mind. So I talked to another participant who told me I was not hallucinating and that they are the twins named Bimal Sinha and Bikas Sinha. Apparently the brothers had heard my name before. We became good friends since then. Being identical twins, at least that's how they looked to me in the early days, I could not distinguish the two brothers for a long time until they grew much older when some changes in their facial features made them relatively identifiable. I have no problem to recognize them now.

I met Bikas again in Cairo, Egypt in September 1991 at the International Statistical Institute (ISI) Meeting. When he found out that I was going to Dhaka before returning to USA, he invited me to visit ISI Kolkata as a visiting scientist. My wife and I spent few days at ISI Kolkata in November 1991. I delivered a series of lectures and met a young assistant professor by the name Dr. Subir K. Bhandari. He requested me to collaborate with him in couple of papers which were later published. It was a great experience and Bikas made sure our stay at the ISI was very comfortable.

Both Bimal and Bikas had visited Ball State at my invitation to give invited talks. Bikas, I believe, visited me twice. It was during Bimal's visit to Ball State in 2002 that I was very surprised when the 'Sagamore of the Wabash Award' which is the highest award of the State of Indiana in USA was awarded to me by the Governor of the State and I was happy that Bimal was present at that ceremony.

Both Bimal and Bikas address me as either Dada or Ali-da. They both have tremendous respect and love for me and I have the same for them. Both of them are proud of their origin in Bangladesh and almost consider Rajshahi University their home university and I have met Bikas there couple of times during my visits to Rajshahi University. I provide below some excerpts from our most recent email correspondences which may explain our relation better. The excerpts from the emails show how much love and respect they have for me.

2/7/21

Dear Ali-da: Here is something for your loving brother Bimal. *[This was a news about the creation of an endowed chair in his name at the University of Maryland Baltimore County.]*

Dear Bimal,

CONGRATULATIONS!!!

Thank you very much for remembering this old statistician and brother to share this great news. It has always been nice to know the Sinha brothers and I am always humbled by both of your respect and love for me. I have been always proud of you both. You have done a lot for the statistics program at UMBC and this is a great recognition for that and also a tribute to your contributions in the field of statistics worldwide. Please visit us if you pass this way. Your Boudi sends her congratulations to you and your family. Love. Ali-da

2/7/21

Hello Dada: Thank you so much for your kind words and so much love for us!

Of course, I remember very well that day/evening - we celebrated together the wonderful news of your extraordinary achievement - Jyoti Sarkar/his wife were also with us during that time. I believe the occasion was - you kindly invited me to visit your dept/give a seminar then in the evening we were together at your place - J/his wife also joined us - had great dinner prepared by Boudi. I also remember after dinner activities, including your playing the harmonium and singing a few Tagore songs in your wonderful melodious voice!

I remember your very kind and gracious offer to Manisha for visiting your univ/dept multiple times at my request. For all these wonderful memories - I am humbled/blessed! You are exactly 10 years ahead of us, but in the picture you look so FIT and SMART and I look like an OLD MAN!!

Love/pranam.

Bimal

2/24/2021

Dear Dada: Our love and respect for you knows no bounds. *[This follows a long narrative of his school and college and university life.]*

Pranam. Bimal

2/26/2021

Dear Bimal,

Thank you for your email. I have a special place in my heart for the twin statisticians Bimal/Bikas. You both are very much revered by statisticians in Bangladesh and especially by Rajshahi University statisticians and I am very proud of you both. I am attaching a picture of both of us during your visit to BSU. You possibly remember you were with me that afternoon when I was surprised to be awarded ‘Sagamore of the Wabash’, the highest honor bestowed by the Governor of Indiana and the highest award of the State of Indiana, USA. I am glad you were with me on that day. I also had a great time at the ISI, Kolkata at the invitation of Bikas when he was the Director. That short visit had also resulted in two joint papers with Subir Bhandari. And you had requested me to include Manisha in my research group. I had brought her to Ball State twice as a visitor and she coauthored with me in many papers over the years. I was sorry to hear about her husband’s demise in an auto accident. May both of you be able to continue with your excellent work. I hope to see both of you soon in person. I am running on 85 and getting a bit weaker. I am otherwise fine.

My deep regards and love for both of you.

Alida

On Feb 27, 2021

Ali-da: If Winter comes, can Spring be far behind? I hope.....you will dig into your photo collection and come up with a replica of this photo with bikas in place of bimal!!! That would be wonderful.

Bks [Bikas]

[I had a picture with Bimal but I was not sure if it was Bimal or Bikas. So I sent the picture to both of them to ascertain who it was and the above was the humorous reply from Bikas.]

*In all these emails ‘we’ everywhere refers to Bimal and Bikas. Their emails reflected the same sentiments.

Bimal and Bikas – I feel very fortunate that my path and your paths crossed. You both are indeed like my own younger brothers. We have many memories of our 32 years of friendship. I sincerely admire both of you for what you have accomplished professionally and what you have done for the field of Statistics. I am very proud of both of you. I am so delighted to see that the *International Journal of Statistical Sciences* published by the Department of Statistics, Rajshahi University is bringing out this special volume in honor of both of you to express their heartfelt gratitude for what you do for them. You never forgot your root, a small village in former East Pakistan, now Bangladesh. Finally, I thank the Department of Statistics, Rajshahi University for honoring these two very loyal scholars Professor Bimal Sinha and Professor Bikas Sinha in this befitting manner.

1. Introduction

The Lomax (Lo) distribution is a right heavy-tail model used in business, actuarial science, biological sciences, engineering, economics, income and wealth inequality, queueing theory, size of cities, and internet traffic modeling. It has been applied to model data obtained from income and wealth (Harris (1968) and Atkinson and Harrison (1978)), firm size (Corbellini et al. (2007)), reliability and life testing (Hassan Al-Ghamdi (2009)), Hirschrelated statistics (Glanzel (2008)). The Lo model is known as a special model form of Pearson type VI distribution and is also considered as a mixture of exponential and gamma distributions. The Lo model belongs to the family of "decreasing" hazard rate function (HRF) and considered as a limiting model of residual lifetimes (Balkema and de Hann (1974) and Chahkandi and Ganjali (2009)). The Lo distribution has been suggested as heavy tailed alternative to the exponential (Exp), Weibull (W) and Gamma distributions (Bryson (1074)). For details about relation between the Lo model and the Burr family and Compound Gamma (CGam) model see Tadikamalla (1980) and Durbey (1970). The main aim of this work is to provide a flexible extension of the Lo distribution using the odd Burr-G (OB-G) family defined by Alizadeh et al. (2017). The new model proved its ability in modeling the "monotonically decreasing", "**J**-hazard rate function", "increasing-constant HRF" and "monotonically increasing". A random variable (RV) Y has the Lomax (Lo) distribution with parameter β_3 if it has cumulative distribution function (CDF) (for $y > 0$) given by

$$W_{\beta_3}(y) = 1 - (1 + y)^{-\frac{1}{\beta_3}}, \tag{1}$$

where $\beta_3 > 0$ refers to the shape parameter. Then the corresponding probability density function (PDF) of (1) is

$$w_{\beta_3}(y) = \frac{1}{\beta_3}(1+y)^{-\frac{1}{\beta_3}-1}. \tag{2}$$

Due to Alizadeh et al. (2017), the CDF of the OB-G family is given by

$$F_{\beta_1,\beta_2,\underline{\xi}}(y) = 1 - \frac{\overline{W}_{\underline{\xi}}(y)^{\beta_1\beta_2}}{\left[W_{\underline{\xi}}(y)^{\beta_1} + \overline{W}_{\underline{\xi}}(y)^{\beta_1}\right]^{\beta_2}}, \tag{3}$$

where $\overline{W}_{\underline{\xi}}(y) = 1 - W_{\underline{\xi}}(y)$. The PDF corresponding to (3) is given by

$$f_{\beta_1,\beta_2,\underline{\xi}}(y) = \frac{\beta_1\beta_2 w_{\underline{\xi}}(y) W_{\underline{\xi}}(y)^{\beta_1-1}\overline{W}_{\underline{\xi}}(y)^{\beta_1\beta_2-1}}{\left[W_{\underline{\xi}}(y)^{\beta_1} + \overline{W}_{\underline{\xi}}(y)^{\beta_1}\right]^{1+\beta_2}}. \tag{4}$$

For $\beta_2 = 1$, the OB-G family reduces to the Odd G (O-G) family (see Gleaton and Lynch (2006)). For $\beta_1 = 1$, the OB-G family reduces to the Proportional Reversed Hazard Rate family (PRHR) (see Gupta and Gupta (2007)). The odd Burr Lomax (OBLo) CDF is given by

$$F_{\underline{\Psi}}(y)|_{(\underline{\Psi}=\beta_1,\beta_2,\beta_3)} = 1 - \frac{(1+y)^{-\beta^*}}{\left\{\left[1-(1+y)^{-\frac{1}{\beta_3}}\right]^{\beta_1} + (1+y)^{-\frac{\beta_1}{\beta_3}}\right\}^{\beta_2}}, \tag{5}$$

where $\beta^* = \frac{\beta_1\beta_2}{\beta_3}$. For $\beta_2 = 1$, the OBLo reduces to the OLo. For $\beta_1 = 1$, the OBLo reduces to the PRHRLo. The PDF corresponding to (5) is given by

$$f_{\underline{\Psi}}(y) = \beta^* \frac{(1+y)^{-(\beta^*+1)}\left[1-(1+y)^{-\frac{1}{\beta_3}}\right]^{\beta_1-1}}{\left\{\left[1-(1+y)^{-\frac{1}{\beta_3}}\right]^{\beta_1} + (1+y)^{-\beta^*}\right\}^{1+\beta_2}}. \tag{6}$$

The HRF for the new model can be derived from $f_{\underline{\Psi}}(y)/\left[1-F_{\underline{\Psi}}(y)\right]$. Many useful Lo extensions can be found in Tahir et al. (2015) (Weibull Lomax distribution), Cordeiro et al. (2018) (the one parameter Lomax system of densities), Altun et al. (2018a) (Odd log-logistic Lomax), Altun et al. (2018a) (Zografos-Balakrishnan Lomax distribution), Elbiely and Yousof (2018) (Weibull generalized Lomax, Rayleigh generalized Lomax and Exponential generalized Lomax distributions), Yousof et al. (2019) (Topp Leone Poisson Lomax distribution), Goual and Yousof (2019) (Lomax inverse Rayleigh), Yousof et al. (2019b) (the Topp-Leone generated Lomax model),Gad et al. (2019) (Burr type XII Lomax, Lomax Burr type XII and Lomax Lomax distributions), Yousof et al. (2019a) (new zero-truncated version of the Poisson Lomax distribution), Yadav et al. (2020) (Topp Leone Lomax distribution), Ibrahim and Yousof (2020) (Poisson Burr X generalized Lomax and Poisson Rayleigh generalized Lomax distributions) and Elsayed Yousof (2021) (extended Poisson Generalized Lomax distribution).

For illustrating the flexibility of the new density and its corresponding HRF we presented Figure 1 (all figures are listed in Appendix A). Figure 1 (left plot) gives some PDF shapes. Figure 2 (right plot) gives some HRF shapes. Based on Figure 1 (left plot) the PDF of the OBLo model can be asymmetric heavy tail right skewed PDF and symmetric PDF. Based on Figure 2 (right plot) the OBLo HRF can be "decreasing" ($\beta_1 = 0.05,\ \beta_2 =,\ \beta_3 = 1$), "**J**-shape" ($\beta_1 = 50,\ \beta_2 = 0.2,\ \beta_3 = 1$), "increasing-constant" ($\beta_1 = 1.45,\ \beta_2 = 1.45,\ \beta_3 = 3$) and "increasing" ($\beta_1 = 5,\ \beta_2 = 5,\ \beta_3 = 1.5$).

The OBLx model could be useful in modeling the asymmetric monotonically increasing hazard rate real data sets as illustrated in Figure 7 (bottom left panel) and Figure 8 (bottom left panel), the real data sets which have no extremes as shown Figure 7 (top right panel) and Figure 8 (top right panel) the real data sets for which their Kernel density is semi-symmetric and bimodal as shown in Figure 7 (bottom right panel) and Figure 8 (bottom right panel). The OBLx model proved its wide applicability in modeling against common Lomax extensions. In modeling of the failure times data, the OBLx model is compared with many well-known Lomax extensions such as the exponentiated Lomax extension, the odd log-logistic Lomax extension, the transmuted Topp-Leone Lo extension, the Kumaraswamy Lo extension, Gamma Lo extension, special generalized mixture Lo extension, the Burr Hatke Lo extension and the proportional reversed hazard rate Lo extension under the consistent-information criteria, Akaike information criteria, Bayesian information criteria and Hannan-Quinn information criteria. In statistical modeling of the service times, the OBLx model is compared with many well-known Lomax extensions such as the exponentiated Lomax extension, the odd log-logistic Lomax extension, the transmuted Topp-Leone Lo extension, the Kumaraswamy Lo extension, Gamma Lo extension, special generalized mixture Lo extension, the Burr Hatke Lo extension and the proportional reversed hazard rate Lo extension under the consistent-information criteria, Akaike information criteria, Bayesian information criteria and Hannan-Quinn information criteria. Additionally, we derived some new bivariate OBLx (BOBLx) via Farlie Gumbel Morgenstern (FGM) copula, modified Farlie Gumbel Morgenstern (FGM) copula, Renyi's entropy and Clayton copula. The Multivariate OBLx (MOBLx) type is also presented using the Clayton copula. However, future works could be allocated to study these new models.

2. Mathematical properties

2.1. Asymptotics and quantile function

In mathematical analysis, the asymptotic analysis is used for describing the limiting behavior of some functions. Asymptotic derivations for the CDF, PDF and HRF can be obtained for the new model. The asymptotics of the CDF, PDF and HRF as $y \to 0$ are given by

$$F_{\underline{\Psi}}(y) \sim \beta_2 \left[1-(1+y)^{-\frac{1}{\beta_3}}\right]^{\beta_1} |_{y\to 0},$$

$$f_{\underline{\Psi}}(y) \sim \beta_1\beta_2 \frac{1}{\beta_3}(1+y)^{-\left(\frac{1}{\beta_3}+1\right)} \left[1-(1+y)^{-\frac{1}{\beta_3}}\right]^{\beta_1-1} |_{y\to 0},$$

and

$$h_{\underline{\Psi}}(y) \sim \beta^{*}(1+y)^{-\left(\frac{1}{\beta_3}+1\right)} \left[1-(1+y)^{-\frac{1}{\beta_3}}\right]^{\beta_1-1} |_{y\to 0}.$$

The asymptotics of CDF, PDF and HRF as $y \to \infty$ are given by

$$1-F_{\underline{\Psi}}(y) \sim \beta_1^{\beta_2}(1+y)^{-\frac{\beta_2}{\beta_3}}|_{y\to\infty},$$

$$f_{\underline{\Psi}}(y) \sim \beta_2\beta_1^{\beta_2}\frac{1}{\beta_3}(1+y)^{-\frac{\beta_1}{\beta_3}-1}|_{y\to\infty}$$

and

$$h_{\underline{\Psi}}(y) \sim \beta_2\frac{1}{\beta_3}(1+y)^{-1}|_{y\to\infty}.$$

For simulation of this new model, we obtain the quantile function (QF) of Y (by inverting (5)), say $y_u = F^{-1}(u)$, as

$$y_u = \left\{ \left[1 - \frac{\left(1 - u_*^{\frac{1}{\beta_2}}\right)^{\frac{1}{\beta_1}}}{u_*^{\frac{1}{\beta_1 \beta_2}} + \left(1 - u_*^{\frac{1}{\beta_2}}\right)^{\frac{1}{\beta_1}}} \right]^{-\beta_3} - 1 \right\} | u_* = 1 - u, \tag{7}$$

Equation (7) is used for simulating the new model.

2.2. Useful representations

Due to Alizadeh et al. (2017), the PDF in (6) can be expressed as

$$f(y) = \sum_{\nu=0}^{\infty} \varsigma_\nu \, w_{1+\nu,\beta_3}(y), \tag{8}$$

where

$$\varsigma_\nu = \frac{\beta_1 \beta_2}{1+\nu} \sum_{i_1, i_2 = 0}^{\infty} \sum_{i_3 = \nu}^{\infty} (-1)^{i_2 + i_3 + \nu} \binom{-(1+\beta_2)}{i_1}$$
$$\times \binom{-[\beta_1(1+i_1)+1]}{i_2} \binom{\beta_1(1+i_1)+i_2+1}{i_3} \binom{i_3}{\nu},$$

and $w_{1+\nu,\beta_3}(y)$ is the PDF of the Lo model with power parameter $1+\nu$. By integrating Equation (8), the CDF of Y becomes

$$F(y) = \sum_{\nu=0}^{\infty} \varsigma_\nu \, W_{1+\nu,\beta_3}(y), \tag{9}$$

where $W_{1+\nu,\beta_3}(y)$ is the CDF of the Lo distribution with power parameter $1+\nu$.

2.3. Moments and incomplete moments

The s^{th} ordinary moment of Y is given by

$$\mu'_{s,Y} = E(Y^s) = \int_{-\infty}^{\infty} y^s \, f(y) dy,$$

then we obtain

$$\mu'_{s,Y} = \sum_{\nu=0}^{\infty} \sum_{\varDelta=0}^{s} \varsigma_\nu \, (1+\nu)(-1)^{\varDelta} \binom{s}{\nu} B\big((1+\nu), 1+\beta_3(\varDelta - s)\big)|_{\left(\frac{1}{\beta_3} > s\right)}, \tag{10}$$

where $B(\tau_1,\tau_2)=\int_0^1 \; t^{\tau_1-1}(1-t)^{\tau_2-1}dt$. Setting $s=1,2,3$and 4 in (10), we have

$$E(Y)=\sum_{\nu=0}^{\infty} \; \sum_{\varDelta=0}^{1} \varsigma_\nu \, (1+\nu)(-1)^{\varDelta}\binom{1}{\varDelta} B\big((1+\nu),1+\beta_3(\varDelta-1)\big)|_{\left(\frac{1}{\beta_3}>1\right)},$$

$$E(Y^2)=\sum_{\nu=0}^{\infty} \; \sum_{\varDelta=0}^{2} \varsigma_\nu \, (1+\nu)(-1)^{\varDelta}\binom{2}{\varDelta} B\big((1+\nu),1+\beta_3(\varDelta-2)\big)|_{\left(\frac{1}{\beta_3}>2\right)},$$

$$E(Y^3)=\sum_{\nu=0}^{\infty} \; \sum_{\varDelta=0}^{3} \varsigma_\nu \, (1+\nu)(-1)^{\varDelta\nu}\binom{3}{\varDelta} B\big((1+\nu),1+\beta_3(\varDelta-3)\big)|_{\left(\frac{1}{\beta_3}>3\right)},$$

and

$$E(Y^4)=\sum_{\nu=0}^{\infty} \; \sum_{\varDelta=0}^{4} \varsigma_\nu \, (1+\nu)(-1)^{\varDelta}\binom{4}{\varDelta} B\big((1+\nu),1+\beta_3(\varDelta-4)\big)|_{\left(\frac{1}{\beta_3}>4\right)},$$

where $E(Y)=\mu_1'$ is the mean of Y. The s^{th} incomplete moment, say $I_s(t)$, of Y can be expressed, from (9), as

$$I_{s,Y}(t)=\int_{-\infty}^{t} y^s f(y)dy=\sum_{\nu=0}^{\infty} \quad \varsigma_\nu \int_{-\infty}^{t} \quad y^s w_{(1+\nu),\beta_3}(y)dy$$

then

$$I_{s,Y}(t)=\textstyle\sum_{\nu=0}^{\infty}\sum_{\varDelta=0}^{s}\varsigma_\nu \, (1+\nu)(-1)^{\varDelta}\binom{s}{\varDelta} B_t\left((1+\nu),1+\frac{1}{\beta_3}(\varDelta-s)\right)|_{\left(\frac{1}{\beta_3}>s\right)}, \quad (11)$$

where $B_y(\tau_1,\tau_2)=\int_0^y t^{\tau_1-1}\,(1-t)^{\tau_2-1}dt$. The first incomplete moment given by (11) with $s=1$ is

$$I_{1,Y}(t)=\sum_{\nu=0}^{\infty}\sum_{\varDelta=0}^{1} \varsigma_\nu \, (1+\nu)(-1)^{\varDelta}\binom{1}{\varDelta} B_t\big((1+\nu),1+\beta_3(\varDelta-1)\big)|_{\left(\frac{1}{\beta_3}>1\right)}.$$

The index of dispersion IxD is the ratio of variance and mean and can be derived as ID(Y) $=\mu_{2,Y}/\mu_{1,Y}'$. It is a measure used to quantify whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model. Figure 2 gives some three-dimensional skewness plots for parameter β_3. Figure 3 shows some three-dimensional kurtosis plots for parameter β_3. Figures 2 and 3 illustrate the wide flexibility of the skewness and the kurtosis of the OBLo model which helps statisticians in modeling various real data sets.

2.4. Some generating functions (GF)

The moment generating function (MGF) can be derived using (8) as

$$M_Y(t)=\sum_{\nu=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\varDelta=0}^{s}\frac{t^s}{s!}\varsigma_\nu(1+\nu)(-1)^{\varDelta}\binom{s}{\varDelta}B\big((1+\nu),1+\beta_3(\varDelta-s)\big)|_{\left(\frac{1}{\beta_3}>s\right)}.$$

The first s derivatives of $M_Y(t)$, with respect to t at $t=0$, yield the first s moments about the origin, i.e.,

$$\mu'_{s,Y}=E(Y^s)=\frac{d^s}{dt^s}M_y(t)|_{(t=0\text{ and }s=1,2,3,\dots)}.$$

The cumulant generating function CGF is the logarithm of the MGF. Thus, s^{th} cumulant, say $\kappa_{s,Y}$, can be obtained from

$$\kappa_{s,Y}=\frac{d^s}{dt^s}log\left[\sum_{\nu=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\varDelta=0}^{s}\frac{t^s}{s!}\varsigma_\nu(1+\nu)(-1)^{\varDelta}\binom{s}{\varDelta}B\big((1+\nu),1+\beta_3(\varDelta-s)\big)\right]|_{(t=0,\text{ and }s=1,2,3,\dots)}.$$

3. Extensions via Copula

In this section, we derive some new bivariate type OBLo (BOBLo) models using Farlie Gumbel Morgenstern (FGM) copula (see Morgenstern (1956), Gumbel (1958) and Gumbel (1960)), modified FGM copula (see Rodriguez-Lallena and Ubeda-Flores (2004)) and Clayton copula and Renyi entropy copula (Pougaza and Djafari (2011)). The multivariate OBLo (MvOBLo) type is also presented. However, future works may be allocated to study these new models (see Al-babtain et al. (2020), Yousof et al. (2020a and 2021), Shehata and Yousof (2021a,b) and Ali et al. (2021a,b)). First, we consider the joint CDF of the FGM family, where $C_\nabla(t,d)=td\big(1+\nabla\overline{t}\overline{d}\big)|_{\overline{t}=1-t,\overline{d}=1-d}$ with the marginal functions $t=F_{\underline{\boldsymbol{\Psi}}_1}(y_1)$, $d=F_{\underline{\boldsymbol{\Psi}}_2}(y_2)$, $\nabla\in(-1,1)$ is a dependence parameter and for every $t,d\in(0,1)$, $C(t,0)=C(0,d)=0$ which is "grounded minimum" and $C(t,1)=t$ and $C(1,d)=d$ which is "grounded maximum", $C(t_1,d_1)+C(t_2,d_2)-C(t_1,d_2)-C(t_2,d_1)\geq 0$.

3.1. BOBLo type via FGM copula

A copula is continuous in t and d; actually, it satisfies the stronger Lipschitz condition, where

$$|C(t_2, d_2) - C(t_1, d_1)| \leq |t_2 - t_1| + |d_2 - d_1|.$$

For $0 \leq t_1 \leq t_2 \leq 1$ and $0 \leq d_1 \leq d_2 \leq 1$, we have

$$Pr(t_1 \leq t \leq t_2, d_1 \leq d \leq d_2) = C(t_1, d_1) + C(t_2, d_2) - C(t_1, d_2) - C(t_2, d_1) \geq 0.$$

Then, setting $\bar{t} = 1 - F_{\underline{\Psi}_1}(y_1)|_{[\bar{t}=(1-t)\in(0,1)]}$ and $\overline{d} = 1 - F_{\underline{\Psi}_2}(y_2)|_{[\overline{d}=(1-d)\in(0,1)]}$, we can esaily obtain the joint CDF of the FGM family. The joint PDF can then be derived from
$c_{\nabla}(t, d) = 1 + \nabla t^* d^*|_{(t^*=1-2t \text{ and } d^*=1-2d)}$ or from
$c(y_1, y_2) = C\left(F_{\underline{\Psi}_1}(y_1), F_{\underline{\Psi}_2}(y_2)\right) f_{\underline{\Psi}_1}(y_1) f_{\underline{\Psi}_2}(y_2)$.

3.2. BOBLo type via modified FGM copula

The modified FGM copula is defined as $C_{\nabla}(t, d) = td[1 + \nabla A(t)B(d)]|_{\nabla\in(-1,1)}$ or $C_{\nabla}(t, d) = td + \nabla Z_t M_d|_{\nabla\in(-1,1)}$, where $Z_t = tA(t)$, and $M_d = dB(d)$ and $A(t)$ and $B(d)$ are two continuous functions on $(0,1)$ with $A(0) = A(1) = B(0) = B(1) = 0$. Let

$$\alpha_1 = inf\left\{Z_t: \frac{\partial}{\partial t} Z_t|_{\sigma_1}\right\} < 0, \alpha_2 = sup\left\{Z_t: \frac{\partial}{\partial t} Z_t|_{\sigma_1}\right\} < 0,$$

$$\theta_1 = inf\left\{M_d: \frac{\partial}{\partial d} M_d|_{\sigma_2}\right\} > 0, \theta_2 = sup\left\{M_d: \frac{\partial}{\partial d} M_d|_{\sigma_2}\right\} > 0.$$

Then, $1 \leq min(\alpha_1\alpha_2, \theta_1\theta_2) < \infty$, where $t\frac{\partial}{\partial t}A(t) = \frac{\partial}{\partial t}Z_t - A(t)$,

$$\sigma_1 = \left\{t: t \in (0,1)|\frac{\partial}{\partial t}Z_t \quad \text{exists}\right\} \text{ and } \sigma_2 = \left\{d: d \in (0,1)|\frac{\partial}{\partial d}M_d \quad \text{exists}\right\}.$$

- Type-I

Consider the following functional form for both Z_t and M_d where $Z_t = t\left[1 - F_{\underline{\Psi}_1}(t)\right]$ and $M_d = d\left[1 - F_{\underline{\Psi}_2}(d)\right]$. Then, the BOBLo-FGM (Type-I) can be derived from $C_{\nabla}(t, d) = td + \nabla Z_t M_d|_{\nabla\in(-1,1)}$.

- Type-II

Let $A(t)^*$ and $B(d)^*$ be two functional forms which satisfy all the conditions stated earlier where $A(t)^*|_{(\nabla_1>0)} = t^{\nabla_1}(1-t)^{1-\nabla_1}$ and $B(d)^*|_{(\nabla_2>0)} = d^{\nabla_2}(1-d)^{1-\nabla_2}$. Then, the corresponding BOBLo-FGM **(**Type-II**)** can be derived from $C_{\nabla,\nabla_1,\nabla_2}(t,d) = td[1+\nabla A(t)^*\, B(d)^*]$.

- Type-III

Let $Z = \overline{t}\big[log\big(1+\overline{t}\big)\big]|_{\overline{t}=1-t}$ and $M = \overline{d}[log(1+d)]|_{\overline{d}=1-d}$. In this case, one can also derive a closed form expression for the associated CDF of the BOBLo-FGM (Type-III) from $C_{\nabla}(t,d) = td(1+\nabla Z\, M)$.

- Type-IV

The CDF of the BOBLo-FGM (Type-IV) model can be derived from $C(t,d) = tF^{-1}_{\underline{\boldsymbol{\Psi}}_2}(d) + dF^{-1}_{\underline{\boldsymbol{\Psi}}_1}(t) - F^{-1}_{\underline{\boldsymbol{\Psi}}_1}(t)F^{-1}_{\underline{\boldsymbol{\Psi}}_2}(d)$ where $F^{-1}_{\underline{\boldsymbol{\Psi}}_1}(t)$ and $F^{-1}_{\underline{\boldsymbol{\Psi}}_2}(d)$ can be easily derived (see Ghosh and Ray (2016)).

3.3. BOBLo type via Ali-Mikhail-Haq copula

Under the stronger Lipschitz condition, the joint CDF of the Archimedean Ali-Mikhail-Haq copula can be expressed as

$$C(d_1,d_2) = \frac{1}{1-\nabla\overline{d_1 d_2}}\, d_1 d_2|_{\nabla\in(-1,1)},$$

and the corresponding joint PDF of the Archimedean Ali-Mikhail-Haq copula can be expressed as

$$c(d_1,d_2) = \frac{1}{\left[1-\nabla\overline{d_1 d_2}\right]^2}\left(1-\nabla+2\nabla\frac{d_1 d_2}{1-\nabla\overline{d_1 d_2}}\right)|_{\nabla\in(-1,1)},$$

and setting $\overline{d_1} = 1 - F_{\underline{\boldsymbol{\Psi}}_1}(y_1)$ and $\overline{d_2} = 1 - F_{\underline{\boldsymbol{\Psi}}_2}(y_2)$ we can derive the joint CDF and the joint PDF of the BOBLo type via Ali-Mikhail-Haq copula.

3.4. BOBLo and MvOBLo type via Clayton copula

The Clayton copula can be considered as $C(d_1,d_2) = [(1/d_1)^{\nabla} + (1/d_2)^{\nabla} - 1]^{-\nabla^{-1}}|_{\nabla\in(0,\infty)}$. Setting $d_1 = F_{\underline{\boldsymbol{\Psi}}_1}(t)$ and $d_2 = F_{\underline{\boldsymbol{\Psi}}_2}(x)$, the BOBLo type can be derived from $C(d_1,d_2) = C(F_{\underline{\boldsymbol{\Psi}}_1}(t), F_{\underline{\boldsymbol{\Psi}}_2}(x))$. Similarly, the MvOBLo **(**m -

dimensional extension) from the above can be derived from $C(d_{\boldsymbol{\Psi}}) = (\sum_{\boldsymbol{\Psi}=1}^{m} d_{\boldsymbol{\Psi}}^{-\nabla} + 1 - m)^{-\nabla^{-1}}$.

3.5. BOBLo type via Renyi's entropy copula

Using the theorem of Pougaza and Djafari (2011) where $C(t,d) = y_2 t + y_1 d - y_1 y_2$, the associated BOBLo will be $C(t,d) = C(F_{\underline{V}_1}(y_1), F_{\underline{V}_2}(y_2))$.

4. Estimation

In this Section, non-Bayesian and Bayesian estimation methods are considered. In first subsection, we will consider four non-Bayesian estimation methods such as the maximum likelihood estimation (MLE) method, ordinary least square estimation (OLSE) method, weighted least square estimation (WLSE) method and Kolmogorov estimation (KE) method. In the second subsection, the Bayesian estimation method under the squared error loss function (SELF) is considered.

4.1. Non-Bayesian estimation methods

The MLE

Let $y_1, y_2, \dots, y_n$ be a random sample of size n from the OBLo distribution with parameters β_1, β_2 and β_3. Let $\underline{\boldsymbol{\Psi}}^{\boldsymbol{T}}$ be the 3×1 parameter vector. For determining the MLE of $\underline{\boldsymbol{\Psi}}$, we have the log-likelihood function

$$\ell = \ell(\underline{\boldsymbol{\Psi}}) = (\beta_1 - 1)\sum_{i=1}^{n} \quad log\left[1 - \tau(y_i)^{-\frac{1}{\beta_3}}\right] + n\, log(\beta^*)$$

$$- (\beta^* + 1)\sum_{i=1}^{n} \quad \tau(y_i)$$

$$-(1+\beta_2)\sum_{i=1}^{n} \quad log\left\{\left[1 - \tau(y_i)^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau(y_i)^{-\beta^*}\right\}.$$

The components of the score vector, $U(\underline{\boldsymbol{\Psi}}) = \frac{\partial \ell(\boldsymbol{\Psi})}{\partial \underline{\boldsymbol{\Psi}}} = \left(\frac{\partial \ell(\underline{\boldsymbol{\Psi}})}{\partial \beta_1}, \frac{\partial \ell(\underline{\boldsymbol{\Psi}})}{\partial \beta_2}, \frac{\partial \ell(\underline{\boldsymbol{\Psi}})}{\partial \beta_3}\right)^T$, are available if needed. Setting $U(\beta_1) = U(\beta_2) = U(\beta_3) = 0$ and solving them simultaneously yields the MLE $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 3×3 observed information matrix $J(\underline{\boldsymbol{\Psi}}) = \{\partial^2 \ell(\underline{\boldsymbol{\Psi}})/\partial m \partial w\} \mid (m, w = \beta_1, \beta_2, \beta_3)$.

OLS

Let $F_{\underline{\boldsymbol{\Psi}}}(y_{[i:n]})$ denote the CDF of OBLo model and let $y_{[1:n]} < y_{[2:n]} < \dots\ y_{[n:n]}$ be the n ordered random sample. The OLSEs are obtained upon minimizing

$$OLSE(\underline{\boldsymbol{\Psi}}) = \sum_{i=1}^{n} \left[F_{\underline{\boldsymbol{\Psi}}}(y_{[i:n]}) - c_{i,n}\right]^2,$$

when, we have

$$OLSE(\underline{\boldsymbol{\Psi}}) = \sum_{i=1}^{n} \left(1 - \frac{\tau(y_{[i:n]})^{-\beta^*}}{\left\{\left[1 - \tau(y_{[i:n]})^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau(y_{[i:n]})^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right)^2,$$

where $c_{i,n} = \frac{i}{n+1}$. The LSEs are obtained via solving the following non-linear equations

$$0 = \sum_{i=1}^{n} \left(1 - \frac{\tau(y_{[i:n]})^{-\beta^*}}{\left\{\left[1 - \tau(y_{[i:n]})^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau(y_{[i:n]})^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right) \hbar_{\beta_1}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}),$$

$$0 = \sum_{i=1}^{n} \left(1 - \frac{\tau(y_{[i:n]})^{-\beta^*}}{\left\{\left[1 - \tau(y_{[i:n]})^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau(y_{[i:n]})^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right) \hbar_{\beta_2}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}),$$

$$0 = \sum_{i=1}^{n} \left(1 - \frac{\tau(y_{[i:n]})^{-\beta^*}}{\left\{\left[1 - \tau(y_{[i:n]})^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau(y_{[i:n]})^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right) \hbar_{\beta_3}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}),$$

where $\hbar_{\beta_1}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}) = \partial F_{\underline{\boldsymbol{\Psi}}}\left(y_{[i:n]}\right)/\partial\beta_1$, $\hbar_{\beta_2}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}) = \partial F_{\underline{\boldsymbol{\Psi}}}\left(y_{[i:n]}\right)/\partial\beta_2$, $\hbar_{\beta_3}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}) = \partial F_{\underline{\boldsymbol{\Psi}}}\left(y_{[i:n]}\right)/\partial\beta_3$.

WLSE

The WLSEs are obtained by minimizing the function $WLSE\left(\underline{\boldsymbol{\Psi}}\right)$ WRT $\beta_1, \beta_2, \beta_3$

$$WLSE\left(\underline{\boldsymbol{\Psi}}\right) = \sum_{i=1}^{n} \quad d_{i,n}\left[F_{\underline{\boldsymbol{\Psi}}}\left(y_{[i:n]}\right) - c_{i,n}\right]^2,$$

where $d_{i,n} = [(1+n)^2(2+n)]/[i(1+n-i)]$. The WLSEs are obtained by solving

$$0 = \sum_{i=1}^{n} \quad d_{i,n}\left(1 - \frac{\tau\left(y_{[i:n]}\right)^{-\beta^*}}{\left\{\left[1 - \tau\left(y_{[i:n]}\right)^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau\left(y_{[i:n]}\right)^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right)\hbar_{\beta_1}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}),$$

$$0 = \sum_{i=1}^{n} \quad d_{i,n}\left(1 - \frac{\tau\left(y_{[i:n]}\right)^{-\beta^*}}{\left\{\left[1 - \tau\left(y_{[i:n]}\right)^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau\left(y_{[i:n]}\right)^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right)\hbar_{\beta_2}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}),$$

$$0 = \sum_{i=1}^{n} \quad d_{i,n}\left(1 - \frac{\tau\left(y_{[i:n]}\right)^{-\beta^*}}{\left\{\left[1 - \tau\left(y_{[i:n]}\right)^{-\frac{1}{\beta_3}}\right]^{\beta_1} + \tau\left(y_{[i:n]}\right)^{-\frac{1}{\beta_3}\beta_1}\right\}^{\beta_2}} - c_{i,n}\right)\hbar_{\beta_3}(y_{[i:n]}, \underline{\boldsymbol{\Psi}}),$$

where $\hbar_{\beta_1}(y_{[i:n]}, \underline{\boldsymbol{\Psi}})$, $\hbar_{\beta_2}(y_{[i:n]}, \underline{\boldsymbol{\Psi}})$, $\hbar_{\beta_3}(y_{[i:n]}, \underline{\boldsymbol{\Psi}})$ are defined above.

KE method

The Kolmogorov estimates (KEs) of $\beta_1, \beta_2, \beta_3$ are obtained by maximizing the function

$$KE = KE(\beta_1, \beta_2, \beta_3) = \overset{1 \le i \le n}{max}\left\{\frac{i}{n} - F_{\underline{\boldsymbol{\Psi}}}\left(y_{[i:n]}\right), F_{\underline{\boldsymbol{\Psi}}}\left(y_{[i:n]}\right) - \frac{i-1}{n}\right\}.$$

4.2. Bayesian estimation

Assume the gamma priors of the parameters $\beta_1, \beta_2, \beta_3$ of the following forms

$$\pi_{1;(v_1,\nu_1)}(\beta_1) \sim \text{Gamma}(v_1, \nu_1),$$
$$\pi_{2;(v_2,\nu_2)}(\beta_2) \sim \text{Gamma}(v_2, \nu_2),$$
$$\pi_{3;(v_3,\nu_3)}(\beta_3) \sim \text{Gamma}(v_3, \nu_3).$$

Assume that the parameters are independently distributed. The joint prior distribution can be written as

$$\pi_{(v_i,\nu_i)}(\beta_1, \beta_2, \beta_3) = \frac{\nu_1^{v_1}\nu_2^{v_2}\nu_3^{v_3}\beta_1^{v_1-1}\beta_2^{v_2-1}\beta_3^{v_3-1}}{\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)} exp[-(\beta_1\nu_1 + \beta_2\nu_2 + \beta_3\nu_3)].$$

The posterior distribution $\pi(\beta_1, \beta_2, \beta_3|\underline{Y})$ of the parameters is defined as

$$\pi(\beta_1, \beta_2, \beta_3|\underline{Y}) \propto \text{likelihood}(\underline{\boldsymbol{\Psi}}|\underline{Y}) \times \pi_{(v_i,\nu_i)}(\beta_1, \beta_2, \beta_3).$$

Under squared error loss function, the Bayesian estimators of $\beta_1, \beta_2, \beta_3$ are the means of their marginal posteriors. It is not possible to obtain the Bayesian estimates through the above formulae. So, the numerical approximation are needed. We propose the use of MCMC techniques namely Gibbs sampler and Metropolis Hastings (M-H) algorithm. Since the conditional posteriors of the parameters $\beta_1, \beta_2, \beta_3$ cannot be obtained in any standard forms, therefore, using a hybrid MCMC for drawing samples from the joint posterior of the parameters is suggested and the full conditional posteriors of $\beta_1, \beta_2, \beta_3$ can be easily derived. The simulation algorithm is given by

1) Provide the initial values, say β_1, β_2 and β_3 then at $h^{(\text{th})}$ stage,
2) Using M-H algorithm, generate $\beta_{1(i)} \sim \pi_1\left(\beta_1|_{\beta_{2(i)},\beta_{3(i)}}, \underline{Y}\right)$;
3) Using M-H algorithm, generate $\beta_{2(i)} \sim \pi_2\left(\beta_2|_{\beta_{1(i)},\beta_{3(i)}}, \underline{Y}\right)$;
4) Using M-H algorithm, generate $\beta_{3(i)} \sim \pi_2\left(\beta_3|_{\beta_{1(i)},\beta_{2(i)}}, \underline{Y}\right)$;
5) Repeat steps $2-5$, $M = 100000$ times to get the samples of size M from the corresponding posteriors of interest. Obtain the Bayesian estimates of $\beta_1, \beta_2, \beta_3$ and using the following formulas

$$\hat{\beta}_{1(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} \beta_1^{[h]}$$

$$\hat{\beta}_{2(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} \beta_2^{[h]}$$

$$\hat{\beta}_{3(\text{Bayesian})} = \frac{1}{M - M_0} \sum_{h=M_0+1}^{M} \beta_3^{[h]}$$

respectively, where $M_0(\approx 50000)$ is the burn-in period of the generated MCMC.

5. Simulation studies for comparing estimation methods

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on N= 1000 generated data sets from the OBLo version where $n = 50{,}100{,}150$ and 300 and some differnt combinations of β_1, β_2 and β_3 (see Table 1, all Tables are listed in Appendix B). The estimates are compared in terms of their

- Bias BIAS($\underline{\boldsymbol{\Psi}}$);
- Root mean-standard error RMSE($\underline{\boldsymbol{\Psi}}$);
- The mean of the absolute difference between the theoretical and the estimates "D-abs" and
- The maximum absolute difference between the true parameters and estimates "D-max".

From Tables 2, 3, 4 and 5 we note that:

- The BIAS ($\underline{\boldsymbol{\Psi}}$) tends to zero when n increases which means that all estimators are consistent.
- The RMSE ($\underline{\boldsymbol{\Psi}}$) increases and tends to zero when n increases which means incidence of consistency property.
- For all sample sizes ($n = 50{,}100{,}150$ and 300) and for all combinations, the Bayesian estimates have the smallest RMSE where:

For blend I and n = 50; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:04979; 0:02324; 0:08212).

For blend II and n = 50; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:08462; 0:09023; 0:13619).

For blend III and n = 50; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:12571; 0:08183; 0:09991).

For blend IV and n = 50; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:07443; 0:08300; 0:16073).

For blend I and n = 100; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:03742; 0:02080; 0:07806).

For blend II and n = 100; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:03380; 0:04244; 0:07857).

For blend III and n = 100; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:09183; 0:04195; 0:05440).

For blend IV and n = 100; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:03760; 0:04162; 0:07287).

For blend I and n = 150; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:02840; 0:02150; 0:04967).

For blend II and n = 150; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:02795; 0:03583; 0:05343).

For blend III and n = 150; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:06512; 0:03660; 0:04957).

For blend IV and n = 150; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:03186; 0:06761; 0:05407).

For blend I and n = 300; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:01934; 0:02115; 0:00853).

For blend II and n = 300; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:02931; 0:01906; 0:02763).

For blend III and n = 300; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:02354; 0:03125; 0:04394).

For blend IV and n = 300; the RMSE$_{\text{Bayes}}$ $(\beta_3, \beta_2, \beta_1)$ = (0:02937; 0:06011; 0:02224).

6. Applications for comparing Bayesian and non-Bayesian methods

6.1. Comparing Bayesian and non-Bayesian methods under failure times

The first real data set (data set **I**) represents the data on failure times of 84 aircraft windshield given in Murthy et al. (2004). We consider the Cramér-Von Mises (W^*) and the Anderson-Darling (A^*) statistic. From Table 6, the WLE method is the best method with W^*=0.76775andA^*=0.07763 then MLE method with W^*=0.95305andA^*=0.10143. However, the worst estimation method in modeling failure times is the OLSE method with W*=1.29159andA^*=0.14925.

6.2. Comparing Bayesian and non-Bayesian methods under service times

The second real data set (data set **II**) represents the data on service times of 63 aircraft windshield given in Murthy et al. (2004). Many other useful real life data sets can be found in Aryal et al. (2017), Yousof et al. (2018), Elbiely and Yousof (2018), Ibrahim and Yousof (2020), Yadav et al. (2020), Mansour et al. (2020e), Goual et al. (2020). From Table 7, the WLSE method is the best method with W^*=1.09682 and A^*=0.18030 then the Baysian method with W^*=1.19580 and A^*=0.19713. However, the worst estimation method in modeling failure times is the KE method with W^*=1.75321andA^*=0.28872.

7. Modeling

7.1. Assessment

Graphically and using the biases and mean squared errors (MSEs), we can perform the simulation experiments to assess the finite sample behavior of the MLEs. The assessment was based on N =1000 replication for all $n|_{(n=50,100,\ldots,500)}$. The following algorithm is considered:

1) Generate N =1000 samples of size $n|_{(n=50,100,\ldots,500)}$ from the OBLo distribution using (7);
2) Compute the MLEs for the N =1000 samples,
3) Compute the SEs of the MLEs for the 1000 samples. The standard errors (SEs) were computed by inverting the observed information matrix.
4) Compute the biases and mean squared errors given for $\underline{\boldsymbol{\Psi}} = \beta_1, \beta_2, \beta_3$. We repeated these steps for $n|n = 50,100, \ldots ,500$ and compute biases $\left(\text{Bias}_{\underline{\boldsymbol{\Psi}}}(n)\right)$ and $MSEs(MSE_h(n))$ for $\underline{\boldsymbol{\Psi}} = \beta_1, \beta_2, \beta_3$ and $n|_{(n=50,100,\ldots,500)}$.

Figures 4, 5 and 6 give the biases (left plots) and MSEs (right plots) for the parameters β_1, β_2 and β_3 respectively. The left plots show how the three biases vary as $n \to \infty$. The right plots show how the three MSEs vary as $n \to \infty$. The broken line in red in Figure 4 corresponds to the biases being 0 . From Figures 6, 7 and 8 (left plots), the biases are generally negative and tends to zero as $n \to \infty$. From Figures 4, 5 and 6 (right plots), the MSEs decrease to zero as $n \to \infty$.

7.2. Applications for comparing models

In this section, we provide two real life applications to two real data sets to illustrate the importance and flexibility of the OBLo model. We compare the fit of the OBLo with some well-known competitive models such as Lomax model, exponentiated Lomax extension, beta Lomax extension, gamma Lomax extension, transmuted Topp-Leone Lomax extension, reduced transmuted Topp-Leone Lomax extension, odd log-logistic Lomax extension, reduced odd log-logistic Lomax extension, reduced Burr-Hatke Lomax extension, proportional reversed hazard rate Lomax extension and special generalized mixture Lomax extension (see Table 8).

For checking the normality of the two real data sets, the Quantile-Quantile (Q-Q) plot is provided. For exploring the HRF for real data, the total time test (TTT) plot is sketched. For exploring the initial density shape nonparametrically, the nonparametric kernel density estimation (NKDE) is given. Figures 7 and 8 give the normal Q-Q plots, the box plots, TTT plots and NKDE plots for the two data sets respectively. Based on Figures 7(a) and 8(a), we note that the normality is nearly exists. Based on Figures 7(b) and 8(b), we note that no extreme values were spotted. Based on Figures 7(c) and 8(c), we note that the HRF is "monotonically increasing" for the two data sets. Figures 7(d) and 8(d) show NKDE is bimodal and nearly symmetric.

We estimate the unknown parameters of each model by maximum likelihood using "L-BFGS-B" method and the goodness-of-fit statistics Akaike information criterion (AIC), Consistent AIC (CAIC), Bayesian IC (BIC), Hannan-Quinn IC (HQIC), A^* and W^* are used to compare the five models.

Regarding the failure times data: Table 11 gives the MLEs and standard errors (SEs). Table 10 gives the goodness-of-fits statistics. Regarding the service times data**:** Table 4 gives the MLEs and SEs. Table 5 gives the goodness-of-fits statistics. Based on resuls of Tables 8 and 10, it is noted that the OBLo model has the lowest values of AIC, CAIC, BIC, HQIC, A^* and W^***.** For failure times data: $\hat{\ell} = -134.3584$, AIC=274.7169, CAIC=275.0169, BIC=282.0093, HQIC=277.6484, $A^* = 0.9444$ and $W^* = 0.1005$.For service times data: $\hat{\ell} = -104.4258$, AIC=214.8517, CAIC=215.2584, BIC=221.2811, HQIC=217.3804, $A^* = 1.2820$ and $W^* = 0.2115$.Moreover, other graphical tools are employed for supporting the numerical results of Table 3 and 5. Figures 9 and 10 give the fitted PDF, fitted CDF, probability-probability (P-P) plot and fitted HRF for data set **I** and data set **II,** respectively.

8. Conclusions

A new three-parameter lifetime model called the odd Burr Lomax is defined and studied. The density of the OBLo model can be asymmetric heavy tail right skewed density and symmetric density with different useful shapes. The hazard rate function of the OBLo can be "monotonically decreasing", "**J**-hazard rate function", "increasing-constant" and "monotonically increasing". The approach of copula is used for deriving many bivariate odd Burr Lomax type distributions. Bayesian and non-Bayesian estimation methods are considered. Four non-Bayesian estimation methods are considered and compared such as the maximum likelihood estimation method, ordinary least square estimation method, weighted least square estimation method and Kolmogorov estimation method. The Bayesian estimation method is considered under the squared error loss function. We assessed the performance of the log-likelihood estimation method via simulation study. The odd Burr Lomax model could be chosen as the best model among Lomax, exponentiated Lomax, Kumaraswamy Lomax, Macdonald Lomax, beta Lomax, gamma Lomax, odd log-logistic Lomax, reduced odd log-logistic Lomax, reduced Burr-Hatke Lomax, reduced odd Burr Lomax and special generalized mixture Lomax distribution in modeling the "failure times" and the "service times" data sets.

As a future work, we can apply many new useful goodness-of-fit tests for the right censored distributional validation such as the Nikulin-Rao-Robson goodness-of-fit test statistic, modified Nikulin-Rao-Robson goodness-of-fit statistic test, Bagdonavicius-Nikulin goodness-of-fit statistic test, modified Bagdonavicius-Nikulin goodness-of-fit statistic test, to the new odd Burr Lomax model as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Salahet al. (2020), Mansour et al. (2020a,d), Ibrahimet al. (2020), Yadav et al. (2020), Goual and Yousof (2020 and 2021b) and Aidi et al. (2021), among others.

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Appendix A:

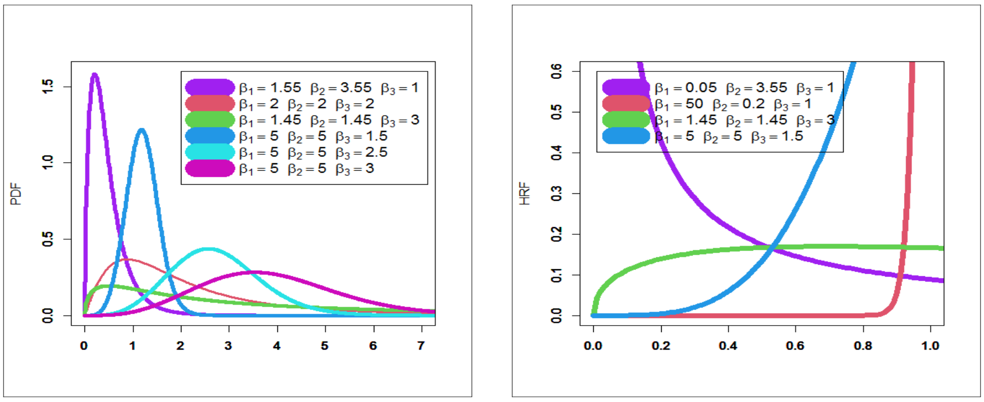


Figure 1: PDF and HRF plots for some selected parameters value.

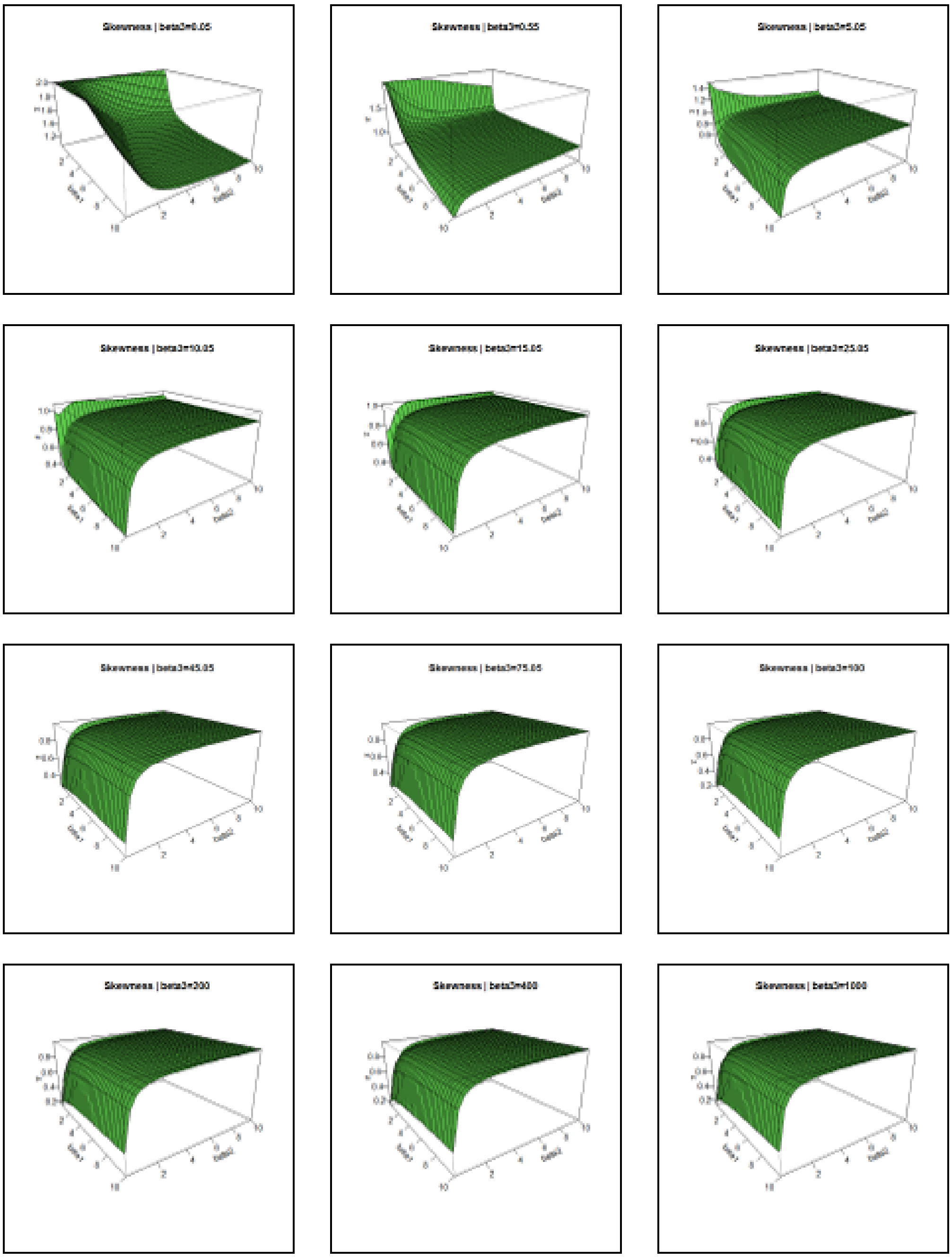


Figure 2: Three dimension skewness plots for parameter β_3

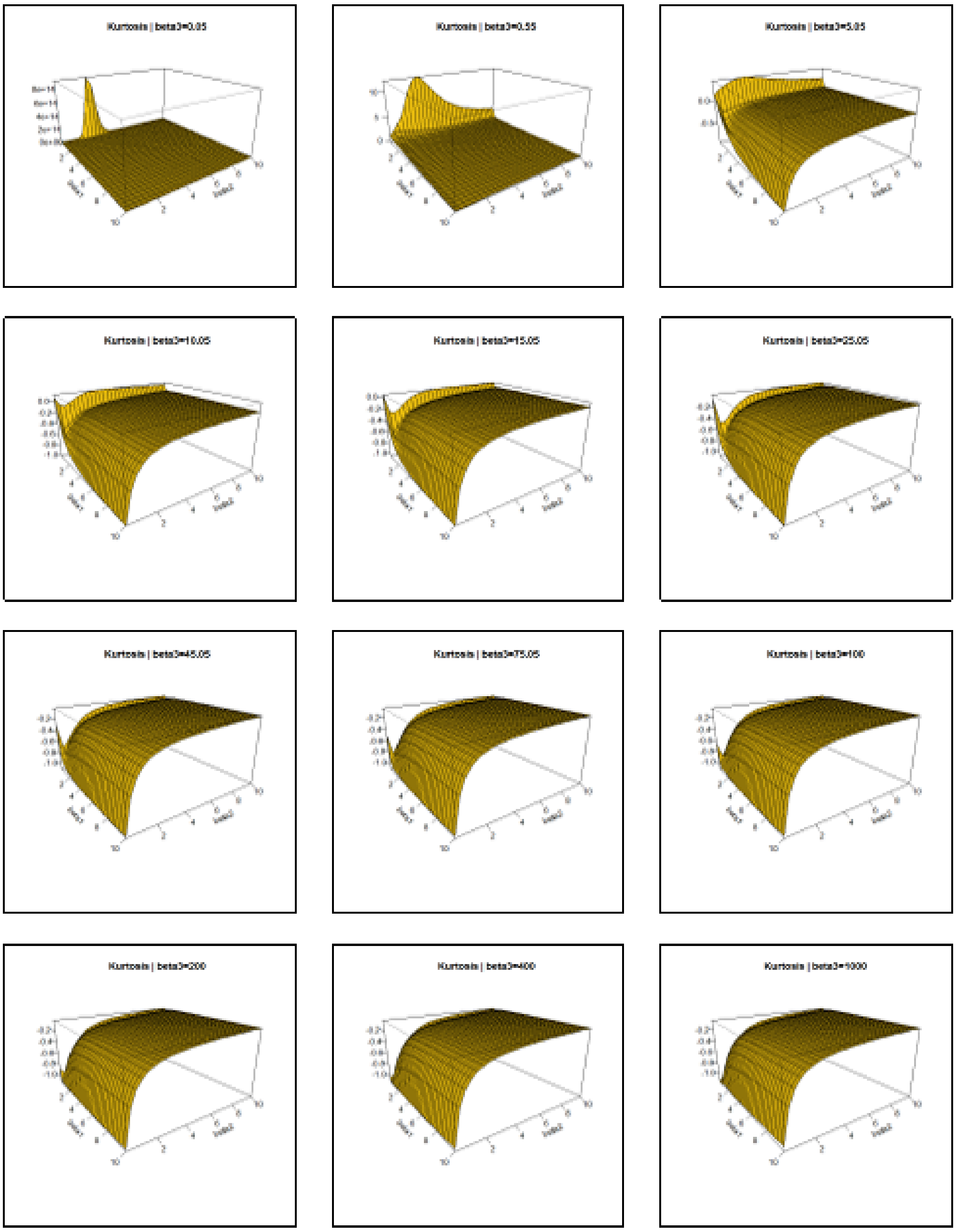


Figure 3: Three dimension kurtosis plots for parameter β_3

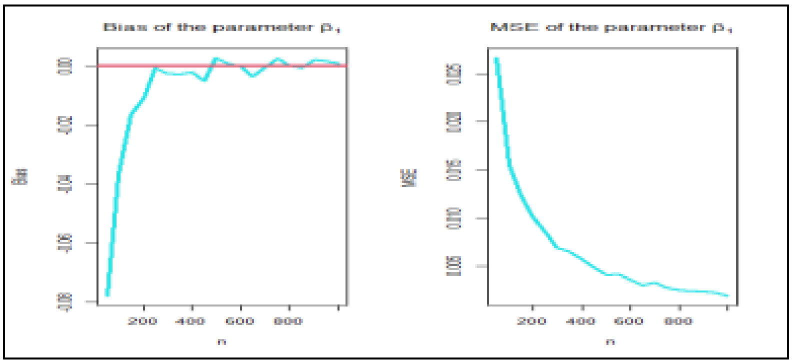


Figure 4: Biases and mean squared errors for the parameter β_1

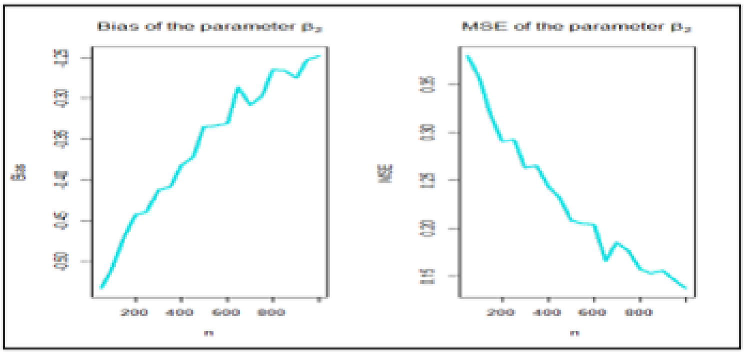


Figure 5: Biases and mean squared errors for the parameter β_2

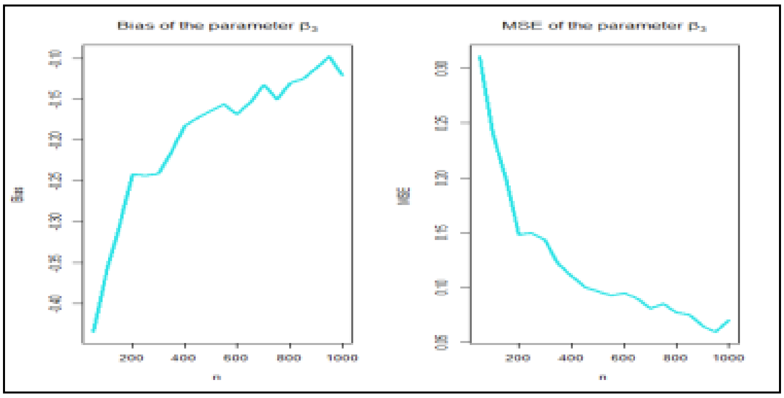
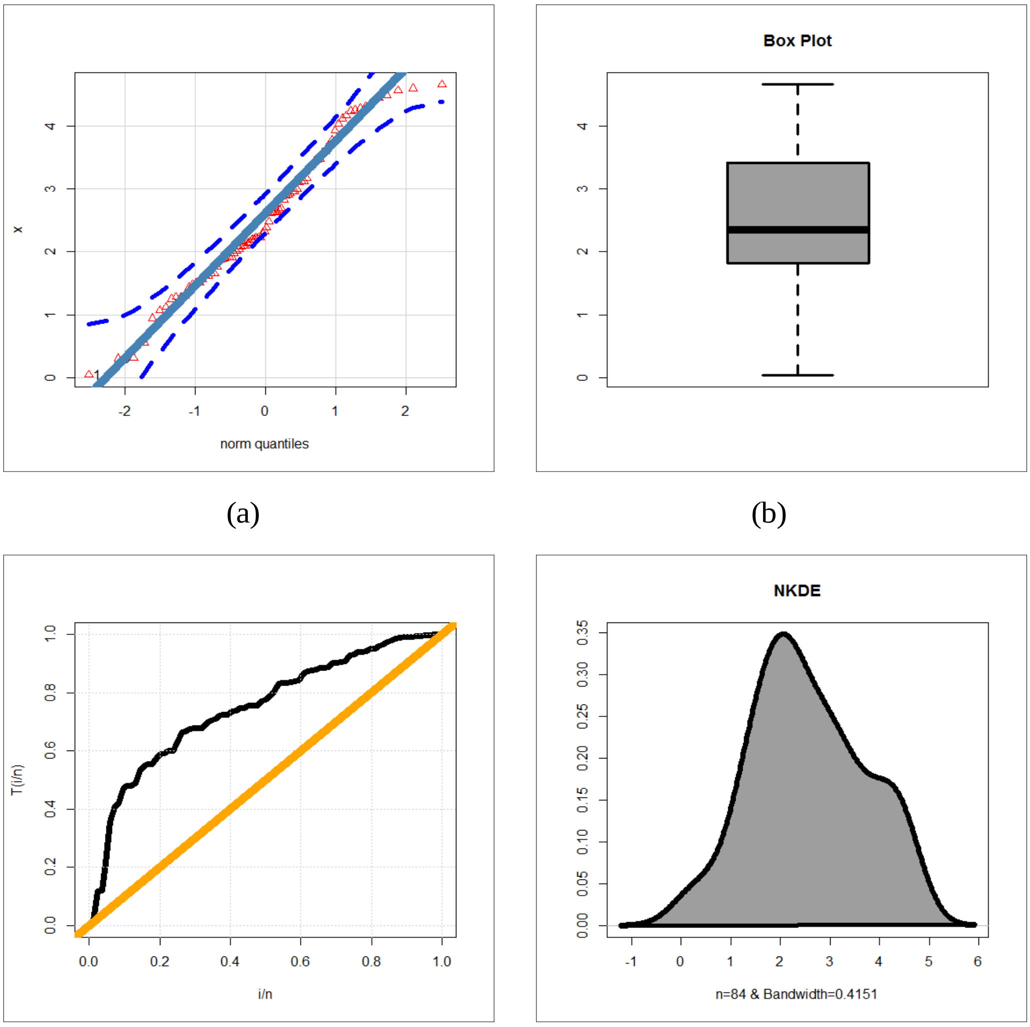


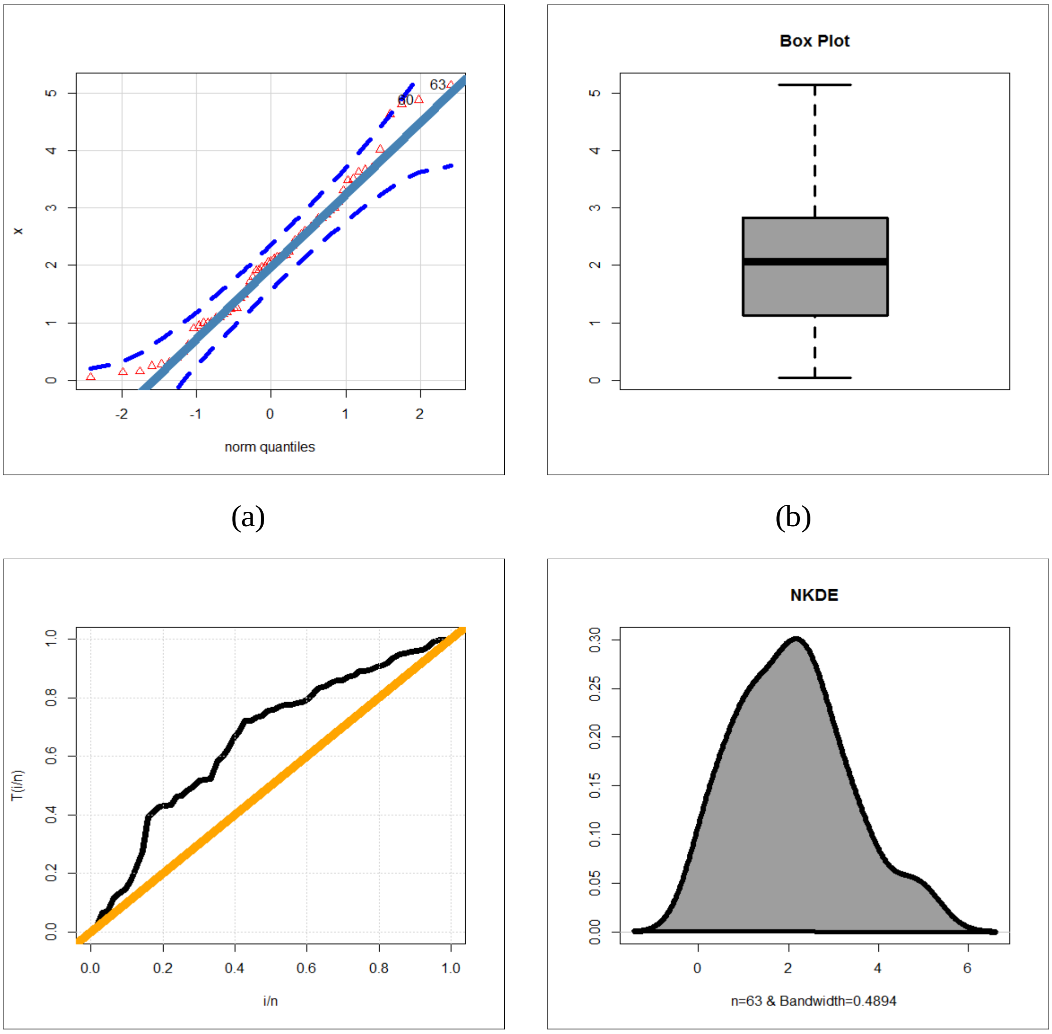
Figure 6: Biases and mean squared errors for the parameter β_3



(a) (b)

(c) (d)

Figure 7: Normal Q-Q plot, box plot, TTT plot and nonparametric KDE for data set **I**.



(a) (b)

(c) (d)

Figure 8: Normal Q-Q plot, box plot, TTT plot and nonparametric KDE for data set **II**.

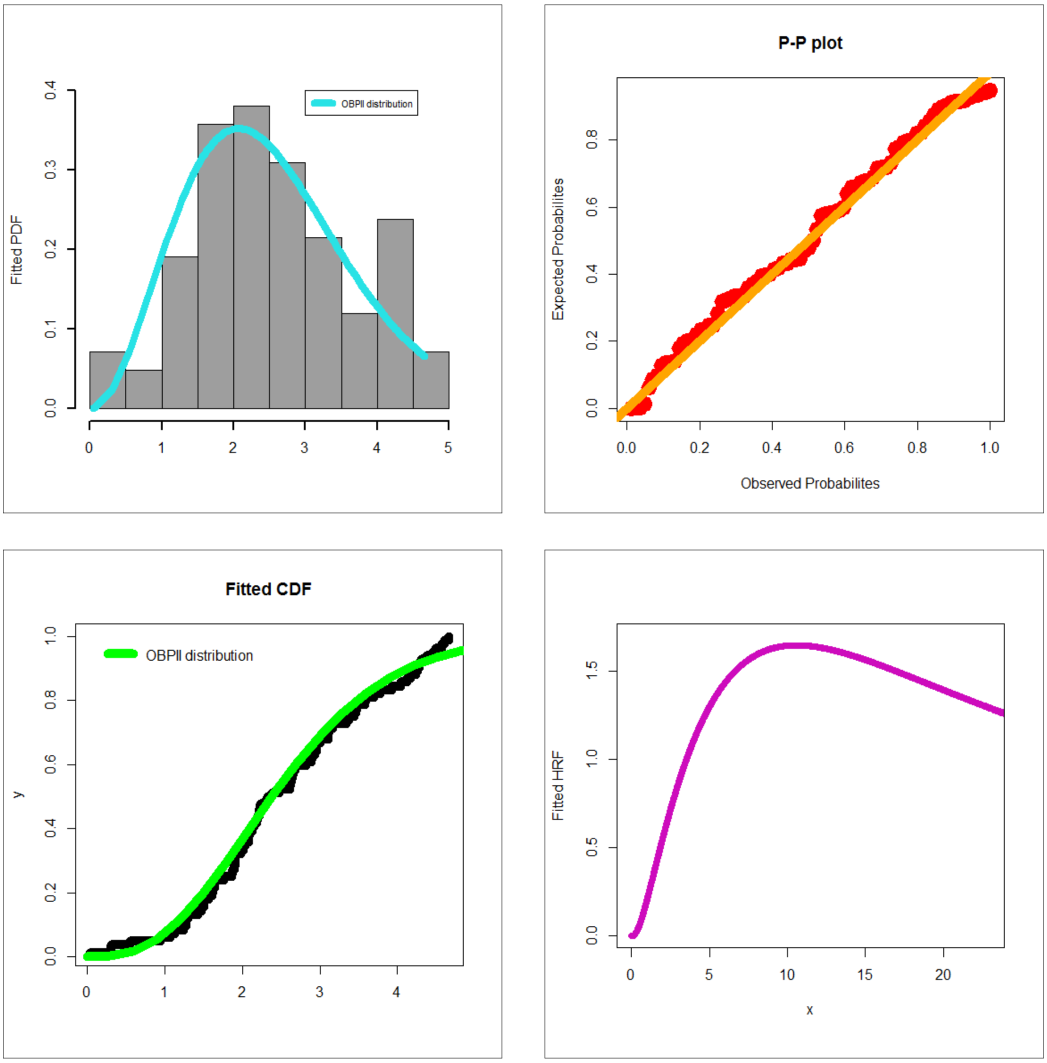


Figure 9: Fitted PDF, P-P plot, fitted CDF and fitted HRF for data set **I**.

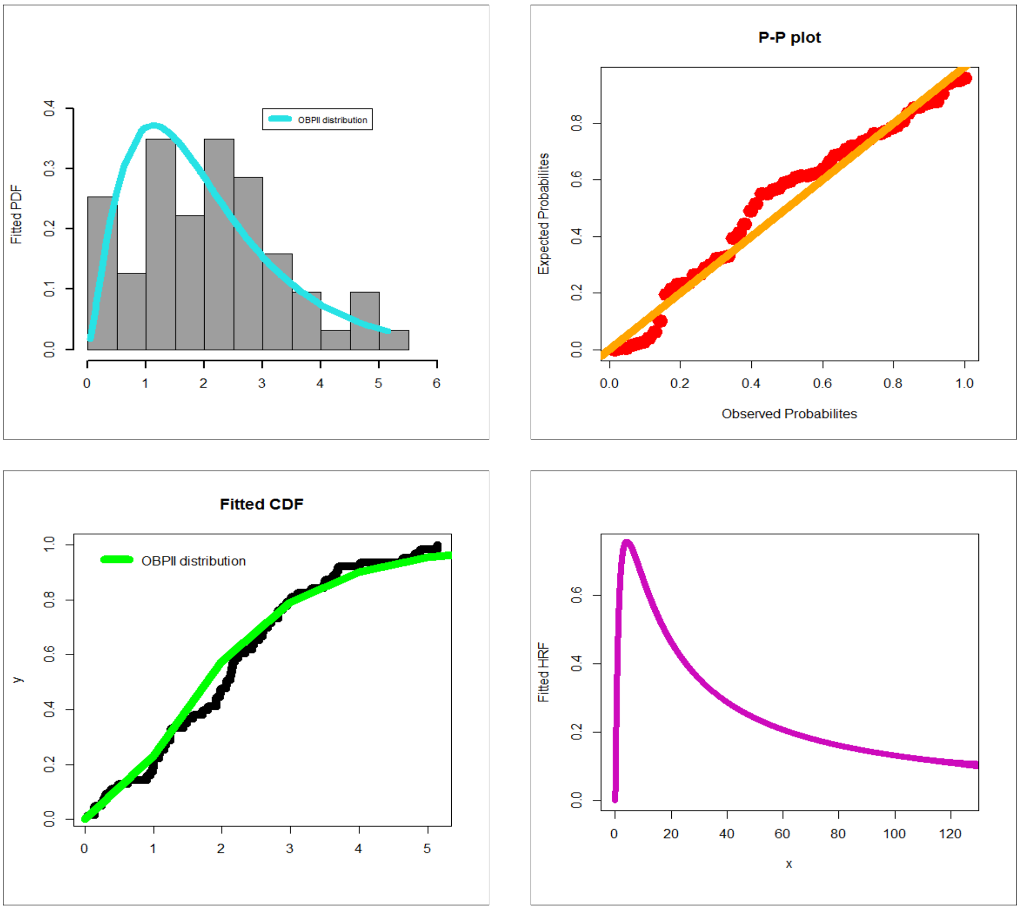


Figure 10: Fitted PDF, P-P plot, fitted CDF and fitted HRF for data set **II**.

Appendix B:

Table 1: Combinations.

Blend	β_1	β_2	β_3
I	2.0	0.5	0.5
II	1.5	0.6	0.8
III	0.9	0.9	1.5
IV	1.8	0.8	1.8

Table 2: Simulation results for blend I.

		BIAS			RMSE			D	
		β_3	β_2	β_1	β_3	β_2	β_1	D-abs	D-max
MLE	50	0.00267	0.00967	0.03580	0.05030	0.07386	0.25600	0.00526	0.00978
OLS		0.00486	0.00350	-0.02645	0.05213	0.08430	0.35576	0.00394	0.00674
WLS		0.00096	0.00760	0.06092	0.04919	0.07523	0.30290	0.00754	0.01391
KE		-0.01004	0.02985	0.05993	0.04941	0.09253	0.40798	0.02652	0.04144
Bayes[1]		0.01258	0.00255	0.08031	0.04979	0.02324	0.08212	0.05810	0.01192
MLE	100	0.00055	0.00589	0.02243	0.03425	0.05154	0.17815	0.00400	0.00717
OLS		0.00195	0.00276	-0.00161	0.03623	0.05842	0.25484	0.00044	0.00065
WLS		0.00145	0.00345	0.05203	0.03672	0.05387	0.21881	0.00493	0.00917
KE		-0.00406	0.01269	0.02394	0.03553	0.06076	0.28596	0.01108	0.01730
Bayes[1]		0.02097	-0.01696	-0.07742	0.03742	0.02080	0.07806	0.02017	0.00003
MLE	150	0.00132	0.00270	0.01344	0.02840	0.04161	0.14668	0.00145	0.00276
OLS		0.00103	0.00197	-0.00606	0.02985	0.04767	0.20766	0.00035	0.00062
WLS		0.00027	0.00325	0.03754	0.03022	0.04371	0.16318	0.00428	0.00795
KE		-0.00376	0.01089	0.01959	0.03041	0.05184	0.23066	0.00966	0.01504
Bayes[1]		0.00408	0.02108	-0.04919	0.02840	0.02150	0.04967	0.02107	0.00003
MLE	300	0.00118	0.00063	0.00553	0.02036	0.02877	0.10189	0.00045	0.00090
OLS		0.00031	0.00146	-0.00053	0.02136	0.03402	0.14806	0.00032	0.00061
WLS		0.00104	0.00060	0.02412	0.02065	0.02910	0.11398	0.00183	0.00325
KE		-0.00111	0.00410	0.00627	0.02094	0.03467	0.16061	0.00333	0.00517
Bayes[1]		0.00078	0.01895	0.12023	0.01934	0.02115	0.00853	0.01146	0.00002

Table 3: Simulation results for blend II.

		BIAS			RMSE			D	
		β_3	β_2	β_1	β_3	β_2	β_1	D-abs	D-max
MLE	50	0.00566	0.01091	0.02735	0.09559	0.09144	0.18560	0.00455	0.00863
OLS		0.00053	0.01321	-0.00505	0.09643	0.09818	0.25218	0.00491	0.00686
WLS		0.00974	0.00511	0.05239	0.10044	0.09525	0.22870	0.00491	0.00815
KE		-0.01139	0.02924	0.02434	0.10157	0.11532	0.31230	0.01833	0.02859
Bayes[1]		0.07336	-0.01790	0.01563	0.08462	0.09023	0.13619	0.07690	0.11186
MLE	100	0.00295	0.00373	0.00870	0.06263	0.05816	0.13081	0.00165	0.00313
OLS		0.00713	0.00104	-0.01490	0.07331	0.07396	0.19181	0.00360	0.00627
WLS		0.00516	0.00210	0.03448	0.07095	0.06470	0.15236	0.00321	0.00573
KE		-0.00618	0.01468	0.01163	0.07252	0.07697	0.21914	0.00939	0.01464
Bayes[1]		0.01977	0.07916	0.05863	0.03380	0.04244	0.07857	0.02599	0.04414
MLE	150	0.00136	0.00346	0.00931	0.05399	0.05038	0.11037	0.00121	0.00233
OLS		0.00181	0.00290	-0.00603	0.05662	0.05655	0.14610	0.00158	0.00277
WLS		0.00265	0.00218	0.02977	0.05683	0.05256	0.12520	0.00316	0.00547
KE		-0.00523	0.01060	0.01777	0.05596	0.05921	0.17388	0.00775	0.01275
Bayes[1]		0.00133	0.00494	-0.11187	0.02795	0.03583	0.05343	0.01169	0.02091
MLE	300	0.00092	0.01110	0.00177	0.03693	0.03401	0.07568	0.00026	0.00051
OLS		0.00255	-0.00010	-0.00657	0.04077	0.04004	0.10371	0.00039	0.00075
WLS		-0.00014	0.00242	0.02024	0.03898	0.03594	0.08470	0.00289	0.00534
KE		-0.00316	0.00575	0.00120	0.04028	0.04125	0.12032	0.00379	0.00562
Bayes[1]		0.01590	0.03487	0.04958	0.02931	0.01906	0.02763	0.01111	0.02083

Table 4: Simulation results for blend III.

	BIAS			RMSE			D	
	β_3	β_2	β_1	β_3	β_2	β_1	D-abs	D-max
MLE 50	-0.01048	0.02301	0.01908	0.21972	0.13362	0.11395	0.00863	0.01582
OLS	0.01487	0.01664	-0.03789	0.26440	0.15351	0.12794	0.00601	0.01044
WLS	0.01969	0.00821	0.02294	0.23133	0.13364	0.12791	0.00351	0.00593
KE	-0.04812	0.05538	0.00393	0.27339	0.17465	0.17387	0.02246	0.03367
Bayes[1]	0.11206	0.01701	-0.07882	0.12571	0.08183	0.09991	0.02631	0.04043
MLE 100	0.00083	0.00836	0.00835	0.16257	0.09302	0.07996	0.00267	0.00513
OLS	0.01433	0.00463	-0.02086	0.19035	0.10596	0.09136	0.00383	0.00694
WLS	0.01522	0.00132	0.02077	0.16767	0.09472	0.09050	0.00313	0.00558
KE	-0.02708	0.02775	0.00197	0.18757	0.11103	0.12064	0.01184	0.01769
Bayes[1]	0.04838	-0.06514	-0.01016	0.09183	0.04195	0.05440	0.02182	0.03743
MLE 150	-0.00030	0.00620	0.00345	0.12945	0.07500	0.06304	0.00187	0.00334
OLS	0.01374	0.00050	-0.01602	0.15171	0.08501	0.07334	0.00358	0.00687
WLS	0.00393	0.00414	0.01816	0.13253	0.07594	0.07568	0.00309	0.00547
KE	-0.00941	0.01343	-0.00097	0.15234	0.08802	0.10355	0.00513	0.00741
Bayes[1]	0.07841	-0.03322	-0.02630	0.06512	0.03660	0.04957	0.01902	0.03679
MLE 300	0.00363	0.00063	0.00126	0.08802	0.05023	0.04453	0.00035	0.00057
OLS	0.00615	0.00062	-0.00543	0.10782	0.05929	0.05351	0.00128	0.00250
WLS	0.00222	0.00183	0.01199	0.09271	0.05320	0.05069	0.00201	0.00338
KE	-0.00662	0.00805	-0.00138	0.11119	0.06310	0.07049	0.00321	0.00455
Bayes[1]	0.00082	0.00831	0.01530	0.02354	0.03125	0.04394	0.00250	0.00503

Table 5: Simulation results for blend IV.

		BIAS			RMSE			D	
		β_3	β_2	β_1	β_3	β_2	β_1	D-abs	D-max
MLE	50	0.01489	0.00700	0.02174	0.16571	0.10872	0.22069	0.00195	0.00343
OLS		0.00400	0.01479	-0.06106	0.17962	0.13645	0.26658	0.00444	0.00866
WLS		0.00693	0.01036	0.05283	0.18171	0.12481	0.26108	0.00521	0.00985
KE		-0.03799	0.04998	0.01203	0.18288	0.15538	0.34980	0.02420	0.03626
Bayes[1]		0.02809	-0.07554	0.14325	0.07443	0.08300	0.16073	0.03076	0.04789
MLE	100	0.00361	0.00600	0.00997	0.12082	0.08149	0.15551	0.00151	0.00296
OLS		0.00684	0.00412	-0.03521	0.12944	0.09419	0.19406	0.00312	0.00594
WLS		0.00755	0.00319	0.04587	0.12944	0.08405	0.18159	0.00371	0.00588
KE		-0.01509	0.02173	0.00675	0.13063	0.10159	0.25389	0.01033	0.01557
Bayes[1]		0.01764	-0.03515	0.01081	0.03760	0.04162	0.07287	0.02707	0.04052
MLE	150	0.00181	0.00455	0.00652	0.10163	0.06686	0.12471	0.00124	0.00237
OLS		0.00729	0.00045	-0.02877	0.10351	0.07556	0.15601	0.00296	0.00519
WLS		0.00673	0.00114	0.03344	0.10472	0.06868	0.15078	0.00305	0.00577
KE		-0.00622	0.01111	-0.00322	0.10385	0.07907	0.19582	0.00479	0.00686
Bayes[1]		-0.1629	-0.04159	0.04551	0.03186	0.06761	0.05407	0.01492	0.02133
MLE	300	0.00185	0.00185	0.00529	0.06910	0.04585	0.09076	0.00055	0.00104
OLS		0.00310	0.00082	-0.00781	0.07340	0.05292	0.11279	0.00086	0.00162
WLS		0.00004	0.00353	0.03029	0.07259	0.06733	0.10194	0.00261	0.00432
KE		-0.00493	0.00691	0.00452	0.07420	0.05545	0.14147	0.00340	0.00528
Bayes[1]		-0.05537	0.05565	0.04286	0.02937	0.06011	0.02224	0.00832	0.01181

Table 6: The values of estimators A^* and W^* under failure data.

Method	$\widehat{\boldsymbol{\beta}}_1$	$\widehat{\boldsymbol{\beta}}_2$	$\widehat{\boldsymbol{\beta}}_3$	$\mathbf{A}^*$	$\mathbf{W}^*$
MLE[2]	3.58192	16.68893	3.49883	0.10143	0.95305
OLS	2.08640	1.91379	3.84051	0.14925	1.29159
WLS[1]	3.29007	17.14852	4.04150	0.07763	0.76775
KE	2.23554	2.79716	4.02079	0.11108	1.03033
Bayes	3.66073	16.63422	3.13238	0.11274	1.03604

Table 7: The values of estimators A^* and W^* under service data.

Method	$\widehat{\boldsymbol{\beta}}_1$	$\widehat{\boldsymbol{\beta}}_2$	$\widehat{\boldsymbol{\beta}}_3$	$\mathbf{A}^*$	$\mathbf{W}^*$
MLE	4.93789	23.12561	2.35843	0.21150	1.28230
OLS	3.29260	7.85586	2.49817	0.20402	1.23969
WLS[1]	3.31252	9.75011	2.64020	0.18030	1.09682
KE	1.79988	1.58316	2.60220	0.28872	1.75321
Bayes[2]	4.67315	22.49038	2.50013	0.19713	1.19580

Table 8: Competitive models.

N	**Model**	**Abbreviation**	**Author**
1	Lomax	Lo	Lomax (1954)
2	Exponentiated Lo	ExpLo	Gupta et al. (1998)
5	Beta Lo	BLo	Lemonte and Cordeiro (2013)
6	Gamma Lo	GamLo	Cordeiro et al. (2015)
7	Transmuted Topp-Leone Lo	TTLLo	Yousof et al. (2017)
8	Reduced TTL Lo	RTTLLo	Yousof et al. (2017)
9	Odd log-logistic Lo	OLLLo	Altun et al. (2018a)
10	Reduced OLL Lo	ROLLLo	Altun et al. (2018a)
11	Reduced Burr-Hatke Lo	RBHLo	Yousof et al. (2018)
13	Proportional reversed hazard rate Lo	PRHRLo	-
14	Special generalized mixture Lo	SGMLo	Chesneau and Yousof (2021)

Table 9: MLEs and SEs for failure times data.

Model	**Estimates**			
OBLo$(\beta_1, \beta_2, \beta_3)$	3.54705	30.65354	4.11768	
	(0.31125)	(53.27151)	(1.66229)	
TTLLo$(v, \beta_1, \beta_2, \beta_3)$	-0.80752	2.47662	(15608.21)	(38628.32)
	(0.139601)	(0.5418)	(1602.366)	(123.9362)
BLo$(v, \beta_1, \beta_2, \beta_3)$	3.60359	33.63866	4.8307011	118.83731
	(0.61872)	(63.71451)	(9.238202)	(428.9271)
PRHRLo$(\beta_1, \beta_2, \beta_3)$	3.74×10^{6}	4.708×10^{-1}	4.5×10^{6}	
	1.03×10^{6}	(0.000012)	37.1468	
RTTLLo$(\beta_1, \beta_2, \beta_3)$	-0.84732	5.52060	1.15682	
	(0.10011)	(1.1848)	(0.0959)	
SGMLo$(\beta_1, \beta_2, \beta_3)$	-1.04×10^{-1}	9.83×10^{6}	1.20×10^{7}	
	(0.12231)	(4843.3)	(501.04)	
ROBLo$(\beta_1, \beta_2, \beta_3)$	3.54792	30.63742	0.24294	
	(0.3141)	(55.8404)	(0.1026)	
OLLLo$(\beta_1, \beta_2, \beta_3)$	2.32640	(7.18×10^{5})	(2.34×10^{6})	
	(2.14×10^{-1})	(1.20×10^{4})	(2.60×10^{1})	
GamLo$(\beta_1, \beta_2, \beta_3)$	3.58761	52001.5	37029.7	
	(0.5133)	(7955.1)	(81.165)	
ExpLo$(\beta_1, \beta_2, \beta_3)$	3.62611	20074.50	26257.7	
	(0.6237)	(2041.83)	(99.742)	
ROLLLo(β_1, β_2)	3.890563	0.57315		
	(0.36523)	(0.0194)		
RBHLo(β_1, β_2)	1080175.1	51367189.2		
	(983309.2)	(232312.2)		
Lo(β_1, β_2)	51425.352	131789.84		
	(5933.494)	(296.1194)		

Table 10: $\hat{\ell}$ and goodness-of-fits statistics for failure times data.

Model	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*
OBLo	-134.3584	274.7169	275.0169	282.0093	277.6484	0.9444	0.1005
OLLLo	-134.4235	274.8470	275.1470	282.1394	277.7785	0.9487	0.1009
ExpLo	-141.3997	288.7994	289.0957	296.1273	291.7469	1.7435	0.2194
GamLo	-138.4042	282.8083	283.1046	290.1363	285.7559	1.3666	0.1618
BLo	-138.7177	285.4354	285.9354	295.2060	289.3654	1.4084	0.1680
Lo	-164.9884	333.9767	334.1230	338.8620	335.9417	1.3976	0.1665
ROLLLo	-142.8452	289.6904	289.8385	294.5520	291.6447	1.9566	0.2554
SGMLo	-143.0874	292.1747	292.4747	299.4672	295.1062	1.3467	0.1578
PRHRLo	-162.8770	331.7540	332.0540	339.0464	334.6855	1.3672	0.1609
RTTLLo	-153.9809	313.9618	314.2618	321.2542	316.8933	3.7527	0.5592
TTLLo	-135.5700	279.1400	279.6464	288.8633	283.0487	1.1257	0.1270
RBHLo	-168.6040	341.2081	341.3562	346.0697	343.1624	1.6711	0.2069

Table 11: MLEs and SEs for service times data.

Model	**Estimates**		
OBLo$(\beta_1, \beta_2, \beta_3)$	2.35846	22.97197	4.92505
	(0.24194)	(41.7776)	(3.2902)
PRHRLo$(\beta_1, \beta_2, \beta_3)$	1.60×10^{6}	3.93×10^{-1}	1.31×10^{6}
	2.02×10^{3}	0.0004×10^{-1}	0.94×10^{6}
RTTLLo$(\beta_1, \beta_2, \beta_3)$	-0.67150	2.74497	1.01238
	(0.18747)	(0.6697)	(0.11412)
ROBLo$(\beta_1, \beta_2, \beta_3)$	2.358364	23.13999	0.20245
	(0.24133)	(41.1819)	(0.1325)
OLLLo$(\beta_1, \beta_2, \beta_3)$	1.664194	6.34×10^{5}	2.02×10^{6}
	(1.82×10^{-1})	(1.73×10^{4})	7.23×10^{6}
ROLLLo(β_1, β_2)	2.372334	0.691091	
	(0.26825)	(0.04492)	
RBHLo(β_1, β_2)	1405552.3	53203423.4	
	(422.005)	(28.5232)	
Lo(β_1, β_2)	99269.782	207019.3	
	(11863.51)	(301.237)	

Table 12: $\hat{\ell}$ and goodness-of-fits statistics for the service times data.

Model	$\hat{\ell}$	**AIC**	**CAIC**	**BIC**	**HQIC**	A^*	W^*
OBLo	-104.4258	214.8517	215.2584	221.2811	217.3804	1.2820	0.2115
OLLLo	-104.9041	215.8082	216.2150	222.2376	218.3369	0.9424	0.1545
ROLLLo	-110.7287	225.4573	225.6573	229.7436	227.1431	2.3472	0.3908
PRHRLo	-109.2986	224.5973	225.004	231.0267	227.126	1.1264	0.1861
RTTLLo	-112.1855	230.3710	230.7778	236.8004	232.8997	2.6875	0.4532
Lo	-109.2988	222.5976	222.7976	226.8839	224.2834	1.1265	0.1861
RBHLo	-112.6005	229.2011	229.4011	233.4873	230.8869	1.3984	0.2316