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Moments of Lower Generalized Order Statistics from Power Function Distribution and its Characterization

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Abstract

Order statistics, record values and several other model of ordered random variables can be viewed as special case of generalized order statistics (gos) [Kamps, 1995]. Pawlas and Szynal (2001) introduced the concept of lower generalized order statistics ($lgos$) to enable a common approach to descending ordered rv's like reversed order statistics and lower record values. In this paper some recurrence relations for single and product moments of lower generalized order statistics from power function distribution [generalized uniform distribution (Proctor, 1987)] have been derived. Further, explicit expressions for single and product moments are obtained and at the end a characterization theorem is given.

Keywords and Phrases: Lower generalized order statistics, moments, recurrence relations, generalized uniform distribution, order statistics, lower record values and characterization.

AMS Classification: Primary 62G30; Secondary 62E10.

1 Introduction

The concept of lower generalized order statistics ($lgos$) is given by Pawlas and Szynal (2001) as follows:

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \text{ for } 1 \leq i \leq n-1.$$

The random variables $X^*(1, n, \tilde{m}, k), X^*(2, n, \tilde{m}, k), \ldots, X^*(n, n, \tilde{m}, k)$ are said to be lower generalized order statistics from an absolutely continuous distribution function $F()$ with the probability density funtion (*pdf*) $f()$, if their joint density function is of the form

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} \big[F(x_i)\big]^{m_i} f(x_i)\Big)\big[F(x_n)\big]^{k-1} f(x_n) \tag{1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

If $m = 0$, $k = 1$, we obtain the joint *pdf* of the order statistics and for $m = -1$, $k \in N$, we get $k - th$ lower record values.

The work of Burkschat *et al*. (2003) may also be seen for dual (lower) generalized order statistics. For characterization of power function distribution through properties of *lgos*, one may refer to Ahsanullah (2005) and Mbah and Ahsanullah (2007).

A random variable X is said to have the generalized uniform (power function) distribution if its *pdf* is of the form

$$f(x) = \frac{\alpha+1}{\theta^{\alpha+1}}\, x^{\alpha}, \quad 0 < x < \theta \tag{2}$$

with *df*

$$F(x) = \Big(\frac{x}{\theta}\Big)^{\alpha+1}, \quad 0 < x < \theta \tag{3}$$

where $\alpha > -1$ is the shape parameter and $\theta > 0$ is the threshold parameter.

Now in view of (2) and (3), we have

$$F(x) = \frac{x}{\alpha+1}\, f(x). \tag{4}$$

The generalize uniform distribution is a uniform distribution at $\alpha = 0$ and is a standard power distribution at $\theta = 1$.

2 Single Moments

Case I: $\gamma_i \neq \gamma_j;\; i \neq j = 1, 2, \ldots, n-1.$

In view of (1) the *pdf* of $r-th$ lower generalized order statistic $X^*(r, n, \tilde{m}, k)$ is

$$f_{X^*(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) \big[F(x)\big]^{\gamma_i - 1} \tag{5}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n.$$

Theorem 2.1: For distribution as given in (2) and $n \in N,\; \tilde{m} \in \mathbb{R},\; k > 0,\; 1 \leq r \leq n,$

$$E\big[X^*(r, n, \tilde{m}, k)\big]^j = \frac{(\alpha+1)\gamma_r}{j + (\alpha+1)\gamma_r}\; E\big[X^*(r-1, n, \tilde{m}, k)\big]^j. \tag{6}$$

Proof: We have Athar *at el*. (2008),

$$E\big[\xi\{X^*(r, n, \tilde{m}, k)\}\big] - E\big[\xi\{X^*(r-1, n, \tilde{m}, k)\}\big]$$

$$= -C_{r-2} \int_{\alpha}^{\beta} \xi'(x) \sum_{i=1}^{r} a_i(r) \big[F(x)\big]^{\gamma_i}\; dx.$$

Let $\xi(x) = x^j$, then

$$E\big[X^*(r, n, \tilde{m}, k)\big]^j - E\big[X^*(r-1, n, \tilde{m}, k)\big]^j$$

$$= -C_{r-2}\; j \int_{\alpha}^{\beta} x^{j-1} \sum_{i=1}^{r} a_i(r) \big[F(x)\big]^{\gamma_i}\; dx.$$

Now in view of (4), we get

$$E\big[X^*(r, n, \tilde{m}, k)\big]^j - E\big[X^*(r-1, n, \tilde{m}, k)\big]^j$$

$$= -\frac{j}{(\alpha+1)\gamma_r} C_{r-1} \int_{\alpha}^{\beta} x^j \sum_{i=1}^{r} a_i(r) \big[F(x)\big]^{\gamma_i - 1} f(x)\; dx$$

and hence the result.

Case II: $m_i = m_j,\ i \neq j = 1, 2, \ldots, n-1.$

The *pdf* of $X^*(r, n, m, k)$ is given as:

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}\big[F(x)\big]^{\gamma_r - 1} f(x) g_m^{r-1}\big(F(x)\big), \tag{7}$$

where,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i \ , \ \ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1} x^{m+1} & , \ \ m \neq -1 \\ -\log x & , \ \ m = -1 \end{cases}$$

and $\quad g_m(x) = h_m(x) - h_m(1), \ \ x \in (0, 1).$

Theorem 2.2: For distribution as given in (2) and $n \in N, \ m \in \Re, \ k > 0, \ 1 \leq r \leq n,$

$$E\big[X^*(r, n, m, k)\big]^j = \frac{(\alpha+1)\gamma_r}{j + (\alpha+1)\gamma_r} \ E\big[X^*(r-1, n, m, k)\big]^j, \tag{8}$$

$$E\big[X^*(r, n, m, k)\big]^j = \theta^j \ (\alpha+1)^r \ \prod_{i=1}^{r} \frac{\gamma_i}{j + \gamma_i(\alpha+1)}. \tag{9}$$

Proof: (8) can be established in view of Athar *at el*. (2008) and (4) on the lines of Theorem 2.1.

Since $X^*(0, n, m, k) = \theta$, the maximum of X in the power function distribution, we have

$$E\big[X^*(r, n, m, k)\big]^j = \frac{(\alpha+1)\gamma_1}{j + (\alpha+1)\gamma_1} \ \theta^j. \tag{10}$$

(9) can be established by writing (8) recursively and using (10) as initial value.

Remark 2.1: Recurrence relation for single moments of order statistics (at $m = 0, k = 1$) is

$$E(X^j_{n-r+1:n}) = \frac{(\alpha+1)(n-r+1)}{j + (\alpha+1)(n-r+1)} \ E(X^j_{n-r+2:n})$$

$$= \theta^j \ (\alpha+1)^r \ \prod_{i=1}^{r} \frac{(n-i+1)}{j+(n-i+1)(\alpha+1)}.$$

Replacing $(n-r+1)$ by $(r-1)$, we have

$$E(X_{r:n}^j) = \frac{j+(\alpha+1)(r-1)}{(\alpha+1)(r-1)} \ E(X_{r-1:n}^j)$$

as obtained by Malik (1967),

or,

$$E(X_{r:n}^j) = \frac{n(\alpha+1)}{n(\alpha+1)+j} \ E(X_{r-1:n-1}^j)$$

as obtained by Khan *el al*. (1983).

Remark 2.2: Recurrence relation for single moments of $k-th$ lower record will be

$$E(X_r^{(k)})^j = \frac{(\alpha+1)k}{j+1+(\alpha+1)k} \ E(X_{r-1}^{(k)})^j$$

as obtained by Bieniek and Szynal (2002),

and

$$E(X_r^{(k)})^j = \theta^j \Big(\frac{(\alpha+1)k}{j+(\alpha+1)k}\Big)^r.$$

3 Product Moments

Case I: $\gamma_i \neq \gamma_j; \ i \neq j = 1, 2, \ldots, n-1$

The joint probability density function (pdf) of $r-th$ and $s-th$ lower generalized order statistics is

$$f_{X^*(r,n,\tilde{m},k),X^*(s,n,\tilde{m},k)}(x,y) = C_{s-1}\Big(\sum_{i=r+1}^{s} a_i^{(r)}(s)\Big[\frac{F(y)}{F(x)}\Big]^{\gamma_i}\Big)$$

$$\times\Big(\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i}\Big)\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}, \quad \alpha \leq y < x \leq \beta. \qquad (11)$$

where,

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s \leq n.$$

Theorem 3.1: For distribution as given in (2). Fix a positive integer k and for $n \in N,\ \tilde{m} \in \mathbb{R},\ 1 \le r < s \le n$,

$$E\big[(X^*(r,n,\tilde{m},k))^i.(X^*(s,n,\tilde{m},k))^j\big]$$

$$= \frac{(\alpha+1)\gamma_s}{j+(\alpha+1)\gamma_s}\ E\big[(X^*(r,n,\tilde{m},k))^i.(X^*(s-1,n,\tilde{m},k))^j\big]. \tag{12}$$

Proof: We have Athar *at el*. (2008),

$$E\big[\xi\{X^*(r,n,\tilde{m},k),X^*(s,n,\tilde{m},k)\}\big]-E\big[\xi\{X^*(r,n,\tilde{m},k),X^*(s-1,n,\tilde{m},k)\}\big]$$

$$= -C_{s-2}\int\int_{\alpha\le y<x\le\beta}\frac{\partial}{\partial y}\ \xi(x,y)\sum_{i=r+1}^{s}a_i^{(r)}(s)\Big[\frac{F(y)}{F(x)}\Big]^{\gamma_i}$$

$$\times\sum_{i=1}^{r}a_i(r)[F(x)]^{\gamma_i}\frac{f(x)}{F(x)}\ dy\ dx. \tag{13}$$

Now consider $\xi(x,y)=\xi_1(x).\xi_2(y)=x^i.y^j$ in (13), then in view of (4) we get

$$E\big[(X^*(r,n,\tilde{m},k))^i.(X^*(s,n,\tilde{m},k))^j\big]-E\big[(X^*(r,n,\tilde{m},k))^i.(X^*(s-1,n,\tilde{m},k))^j\big]$$

$$= -\frac{jC_{s-1}}{\gamma_s(\alpha+1)}\int_0^{\theta}\int_0^{x}x^iy^j\sum_{i=r+1}^{s}a_i^{(r)}(s)\Big[\frac{F(y)}{F(x)}\Big]^{\gamma_i}$$

$$\times\sum_{i=1}^{r}a_i(r)[F(x)]^{\gamma_i}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\ dy\ dx,$$

which leads to (12).

Case II: $m_i = m_j = m;\ i \ne j = 1,2,\ldots,n-1.$

The joint *pdf* of $X^*(r,n,m,k)$ and $X^*(s,n,m,k),\ 1 \le r < s \le n$ is given as

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\ [F(x)]^m f(x)\ g_m^{r-1}\big(F(x)\big)$$

$$\times[h_m\big(F(y)\big)-h_m\big(F(x)\big)]^{s-r-1}f(y)[F(y)]^{\gamma_s-1},\ x>y \tag{14}$$

Theorem 3.2: For distribution as given in (2). Fix a positive integer k and for $n \in N,\ m \in \mathbb{R},\ 1 \leq r < s \leq n,$

$$E\big[(X^*(r,n,m,k))^i.(X^*(s,n,m,k))^j\big]$$

$$= \frac{(\alpha+1)\gamma_s}{j+(\alpha+1)\gamma_s}\ E\big[(X^*(r,n,m,k))^i.(X^*(s-1,n,m,k))^j\big], \tag{15}$$

and

$$E\big[(X^*(r,n,m,k))^i.(X^*(s,n,m,k))^j\big]$$

$$= (\alpha+1)^s\ \theta^{i+j}\ \Big(\prod_{u=1}^{r} \frac{\gamma_u}{(\alpha+1)\gamma_u+i+j}\Big)\Big(\prod_{v=r+1}^{s} \frac{\gamma_v}{(\alpha+1)\gamma_v+j}\Big). \tag{16}$$

Proof: (15) can be proved on the lines of Theorem 3.1. To obtain (16), we write (15) recursively.

Remark 3.1: Recurrence relation for product moments of order statistics (at $m = 0,\ k = 1$) is

$$E[X^i_{n-r+1:n}.X^j_{n-s+1:n}] = \frac{(\alpha+1)(n-s+1)}{j+(\alpha+1)(n-s+1)} E[X^i_{n-r+1:n}.X^j_{n-s+2:n}]$$

$$= (\alpha+1)^s\ \theta^{i+j}\ \Big(\prod_{u=1}^{r} \frac{(n-u+1)}{(\alpha+1)(n-u+1)+i+j}\Big)\Big(\prod_{v=r+1}^{s} \frac{(n-v+1)}{(\alpha+1)(n-v+1)+j}\Big).$$

That is

$$E[X^i_{r:n}.X^j_{s:n}] = \frac{j+(\alpha+1)(s-1)}{(\alpha+1)(s-1)} E[X^i_{r:n}.X^j_{s-1:n}],\ 1 \leq r < s \leq n,\ (s-r) \geq 1.$$

Remark 3.2: Recurrence relation for product moments of $k-th$ record values will be

$$E[(X_r^{(k)})^i.(X_s^{(k)})^j] = \frac{(\alpha+1)k}{j+(\alpha+1)k}\ E[(X_r^{(k)})^i.(X_{s-1}^{(k)})^j]$$

$$= \theta^{i+j}\Big(\frac{(\alpha+1)k}{i+j+(\alpha+1)k}\Big)^r\Big(\frac{(\alpha+1)k}{j+(\alpha+1)k}\Big)^{s-r}.$$

Remark 3.3: At $i = 0$, we obtain recurrence relation for single moments as given in (8) and (9).

4 Characterization

Let $X^*(r,n,m,k)$, $r = 1,2,\ldots,n$ be *lgos*, then the conditional *pdf* of $X^*(s,n,m,k)$ given $X^*(r,n,m,k) = x$, $1 \le r < s \le n$, in view of (7) and (14) is

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[F(x)]^{m-\gamma_r+1}$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y). \qquad (17)$$

Theorem 4.1: Let $X^*(r,n,m,k),\ r = 1,2,\ldots,n$ be *lgos* based on continuous distribution function $F()$. Then for two consecutive values r and $r+1,\ 2 \le r+1 \le s \le n$, the conditional expectation of *lgos* $X^*(s,n,m,k)$ given $X^*(r,n,m,k) = x$, is given by

$$E[X^*(s,n,m,k)|X^*(l,n,m,k) = x] = a_{s|l}x, \quad l = r, r+1 \qquad (18)$$

if and only if X has the *df*

$$F(x) = \left(\frac{x}{\theta}\right)^{\alpha+1}, \quad 0 < x < \theta \qquad (19)$$

where,

$$a_{s|l} = \prod_{i=l+1}^{s} \frac{(\alpha+1)\gamma_i}{1+(\alpha+1)\gamma_i}.$$

proof: We have for $s \ge r+1$,

$$g_{s|r} = E[X^*(s,n,m,k)|X^*(r,n,m,k) = x]$$
$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{1}{(m+1)^{s-r-1}}$$
$$\times \int_{\alpha}^{x} y\Big(\frac{F(y)}{F(x}\Big)^{\gamma_s-1}\Big[1 - \frac{(F(y))^{m+1}}{(F(x))^{m+1}}\Big]^{s-r-1}\frac{f(y)}{F(x)}dy \qquad (20)$$

Let $u = \frac{F(y)}{F(x)} = \left(\frac{y}{x}\right)^{\alpha+1}$, then $y = x\ u^{\frac{1}{(\alpha+1)}}$.

Thus (20) becomes

$$E[X^*(s,n,m,k)|X^*(r,n,m,k) = x]$$
$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!}\frac{1}{(m+1)^{s-r-1}}\int_0^1 x\ u^{\frac{1}{(\alpha+1)}}u^{\gamma_s-1}[1-u^{m+1}]^{s-r-1}du.$$

Set $u^{m+1} = t$ to get

$$E[X^*(s,n,m,k)|X^*(r,n,m,k) = x]$$

$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!} \frac{1}{(m+1)^{s-r}} \int_0^1 x \; t^{\frac{1}{(\alpha+1)(m+1)} + \frac{\gamma_s - 1}{m+1} - \frac{m}{m+1}} [1-t]^{s-r-1} dt$$

$$= a_{s|r} x,$$

where,

$$a_{s|r} = \prod_{i=r+1}^{s} \frac{(\alpha+1)\gamma_i}{1+(\alpha+1)\gamma_i} \qquad \text{[Khan and Alzaid, 2004].}$$

To show that (18) implies (19), we have

$$g_{s|r+1}(x) - g_{s|r}(x) = a_{s|r+1}x - a_{s|r}x = (a_{s|r+1} - a_{s|r})x$$

$$= \frac{a_{s|r}}{(\alpha+1)\gamma_{r+1}} \, x.$$

Therefore,

$$\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{g_{s|r+1}(x) - g_{s|r}(x)} = \frac{(\alpha+1)}{x} \tag{21}$$

and hence

$$\frac{f(x)}{F(x)} = \frac{(\alpha+1)}{x}.$$

Implying that

$$F(x) = \left(\frac{x}{\theta}\right)^{(\alpha+1)}, \quad 0 < x < \theta.$$

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